

SOME REGULARITY THEOREMS FOR CARNOT-CARATHÉODORY METRICS

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1. Introduction

Let M be a smooth connected m -dimensional manifold and Q a smooth q -dimensional distribution on M which is *bracket generating*, i.e., for every $p \in M$ the local sections of Q near p span together with all their commutators the tangent space $T_p M$ of M at p .

A curve φ in M is called *horizontal* if φ is tangent almost everywhere to Q . It is a classical result of Chow that any two points of M can be joined by a horizontal curve (see e.g. [13, 12]). Thus if Q is equipped with a Riemannian metric $\langle \cdot, \cdot \rangle_Q$, then the function $d_c: M \times M \rightarrow \mathbb{R}$, $(p, u) \rightarrow \inf\{\text{length}(\varphi) \mid \varphi \text{ is horizontal and joins } p \text{ to } u\}$ is a distance on M , the *Carnot-Carathéodory metric* induced by $(Q, \langle \cdot, \cdot \rangle_Q)$.

Let $\langle \cdot, \cdot \rangle$ be an extension of $\langle \cdot, \cdot \rangle_Q$ to a Riemannian metric on M , and let dist be the induced distance on M . Then $d_c \geq \text{dist}$, and any rectifiable curve with respect to d_c is rectifiable with respect to dist , hence differentiable almost everywhere and moreover horizontal [13]. Vice versa every horizontal curve is locally rectifiable with respect to d_c ; its d_c -length coincides with its usual length as a curve in $(M, \langle \cdot, \cdot \rangle)$ (see [17]; this also follows from the general theory of length structures in [6]). Thus (M, d_c) is a locally compact *length space* and complete if this is true for (M, dist) .

Let $p \in M$ and $\varepsilon > 0$ be such that the closure of the open d_c -ball B of radius ε around p is compact. Then it follows from the theory of locally compact length spaces [6] that every $u \in M$ with $d_c(p, u) < \varepsilon$ can be joined to p by a minimizing *geodesic* with respect to d_c , i.e., a horizontal curve which realizes locally the d_c -distance of its curve points (this is also proved in [17]). Strichartz showed that if Q satisfies the *strong bracket generating hypothesis* (see [17]), i.e., if TM is generated by Q and $[X, Q]$ for every nonzero local section X of Q , then these geodesics are solutions of a system of Hamilton-Jacobi equations on the cotangent bundle T^*M of M , in particular they are smooth curves. This

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leads to the definition of an exponential map of an open neighborhood of the zero section of $T'M$ onto M ; however its restriction to a fiber of $T'M$ is not of maximal rank at 0.

In this paper we give a different approach to the theory of geodesics. We extend $\langle \cdot, \cdot \rangle_Q$ to a Riemannian metric $\langle \cdot, \cdot \rangle$ on M and consider a variational problem in $(M, \langle \cdot, \cdot \rangle)$. We obtain a simple differential equation for the critical points of this variational problem and show that these critical points are geodesics with respect to d_c , i.e., they are locally minimizing curves (this answers a question in [17]). On the other hand, Bär [1] showed that every geodesic is a critical point; together this gives a complete description of the geodesics.

This leads to the definition of an exponential map \exp_p^c at a given point $p \in M$ which maps an open neighborhood Ω of 0 in T_pM onto an open neighborhood of p in M . We show that \exp_p^c is of maximal rank on an open and dense subset of Ω . However \exp_p^c depends on the choice of the extension of $\langle \cdot, \cdot \rangle_Q$ to a Riemannian metric on M and moreover on the choice of a local trivialization of TM adapted to our situation. If the distribution Q satisfies the strong bracket generating hypothesis, then $(Q, \langle \cdot, \cdot \rangle_Q)$ determines a unique Riemannian metric $\langle \cdot, \cdot \rangle$ on M extending $\langle \cdot, \cdot \rangle_Q$ and thus \exp_p^c only depends on the local trivialization. Moreover every d_c -geodesic emanating from p is uniquely determined by its tangent and the covariant derivative of its tangent at p , i.e., \exp_p^c is defined intrinsically.

As an application of the investigation of geodesics we show that any isometry between manifolds with Carnot-Carathéodory (briefly CC-) metrics is necessarily smooth and clearly commutes with the exponential map. We conclude the paper with an example where the geodesics can easily be computed explicitly.

2. The space of H_1 -curves in M through a given point

Let $p \in M$. We consider the Hilbert manifold $H_1^p(I, U)$ of all continuous, absolutely continuous curves $\varphi: I \rightarrow U$ through $\varphi(0) = p$ with square integrable derivative, where U is a suitable open neighborhood of p .

Fix a Riemannian metric $\langle \cdot, \cdot \rangle$ on M extending $\langle \cdot, \cdot \rangle_Q$. Given $p \in M$ select a local orthonormal basis $\{X^1, \dots, X^q\}$ of Q and a local orthonormal basis $\{X^{q+1}, \dots, X^m\}$ of the $\langle \cdot, \cdot \rangle$ -orthogonal complement Q^\perp of

\mathcal{Q} . The local frame $\{X^1, \dots, X^m\}$, defined on an open d_c -ball U of radius $\rho > 0$ around p , will be called *admissible*. Let $\theta^1, \dots, \theta^m$ be the dual coframe and let $\theta = (\theta^1, \dots, \theta^m)$. θ is a 1-form on U with values in a Euclidean m -space \mathbb{R}^m .

The map Θ , defined on $H_1^p(I, U)$ by $(\Theta\varphi)(t) = \theta\varphi'(t)$, has its image in the Hilbert space $H_0(I, \mathbb{R}^m)$ of square integrable curves in \mathbb{R}^m .

Lemma 2.1. Θ is a diffeomorphism of $H_1^p(I, U)$ onto an open neighborhood of 0 in $H_0(I, \mathbb{R}^m)$.

The proof uses the fact that the Banach-manifold of all continuously differentiable curves in U starting at p is diffeomorphic to an open neighborhood of 0 in the Banach space of continuous curves in $T_pM \sim \mathbb{R}^m$ (see [9]) and a standard completion argument.

There are unique 1-forms θ_j^i on U such that

(a) $\theta_j^i = -\theta_i^j$,

(b) $d\theta^i = \sum_{k=1}^m \theta^k \wedge \theta_k^i$.

(see [16]).

Let $\varphi \in H_1^p(I, U)$ and let X be an element of the tangent space $H_1^p(\varphi)$ of $H_1^p(I, U)$ at φ , i.e., X is a section of TM over φ of class H_1 which vanishes at $p = \varphi(0)$. Denote by $\frac{D}{dt}X$ the covariant derivative of X with respect to the Riemannian connection of $\langle \cdot, \cdot \rangle$. Then

Lemma 2.2.

$$\theta^i \left(\frac{D}{dt}X \right) = \frac{d}{dt}(\theta^i(X)) + \sum_{j=1}^m \theta_j^i(\varphi')\theta^j(X).$$

Lemma 2.2 is well known and can be found in [16].

Write $d\theta = (d\theta^1, \dots, d\theta^m)$; $d\theta$ is a 2-form on U with values in \mathbb{R}^m . As a corollary of 2.2, the differential $d\Theta_\varphi$ of Θ at φ can be computed as follows:

Lemma 2.3. If $x \in H_1^p(\varphi)$, then $d\Theta_\varphi X = \frac{d}{dt}(\theta X) - 2d\theta(\varphi', X)$.

Proof. Let $\Psi: (-\varepsilon, \varepsilon) \times I \rightarrow U$ be a variation of $\varphi = \Psi_0$ with variation vector field $X = \frac{\partial}{\partial s}\Psi|_{s=0}$ such that $\Psi(-\varepsilon, \varepsilon) \times \{0\} = p$. Then

$$(d\Theta_\varphi X)(t) = \frac{\partial}{\partial s}\theta \frac{\partial}{\partial t}\Psi(s, t)|_{s=0},$$

and by Lemma 2.2 the i th component of $(d\Theta_\varphi X)(t)$ equals

$$\theta^i \left(\frac{D}{\partial s} \frac{\partial}{\partial t} \Psi(s, t) \right)_{s=0} - \sum_{j=1}^m \theta_j^i(X)\theta^j(\varphi').$$

Using $\frac{D}{\partial s} \frac{\partial}{\partial t} \Psi(s, t) = \frac{D}{\partial t} \frac{\partial}{\partial s} \Psi(s, t)$ and again Lemma 2.2 for $\frac{D}{\partial t} \frac{\partial}{\partial s} \Psi(s, t)$ we obtain for the i th component of $(d\Theta_\varphi X)(t)$ the value

$$\frac{d}{dt}(\theta^i X)(t) + \sum_{j=1}^m (\theta_j^i(\varphi')\theta^j(X) - \theta_j^i(X)\theta^j(\varphi'))$$

which shows the claim. q.e.d.

Now for every $u \in U$ and $X \in T_u M$ the assignment $Y \rightarrow 2d\theta(X, Y)$ is a linear mapping of $T_u M$ into \mathbb{R}^m . Let $a^*(X)$ be its adjoint with respect to the scalar product $\langle \cdot, \cdot \rangle_u$ on $T_u M$ and the Euclidean scalar product $\langle \cdot, \cdot \rangle$ on \mathbb{R}^m . $a^*(X)$ is a linear map of \mathbb{R}^m into $T_u M$ which satisfies $\langle 2d\theta(X, Y), Z \rangle = \langle Y, a^*(X)Z \rangle$ for all $Y \in T_u M$, $Z \in \mathbb{R}^m$. Moreover the assignment $X \rightarrow a(X) = \theta a^*(X)$ is a smooth 1-form on U with values in the vector space of linear endomorphisms of \mathbb{R}^m . For convenience we will also write $a(X, Z)$ instead of $a(X)Z$.

Remark 2.4. The form a can also be computed as follows: Let b_{jk}^i ($i, j, k = 1, \dots, m$) be the unique smooth functions on U which satisfy $d\theta^i = \frac{1}{2} \sum_{j,k} b_{jk}^i \theta^j \wedge \theta^k$ and $b_{jk}^i = -b_{kj}^i$. Then an easy computation shows $\theta^i a(X, Z) = \sum_{j=1}^m \sum_{k=1}^m b_{ik}^j \theta^k(X) Z_j$ for all $Z = (Z_1, \dots, Z_m) \in \mathbb{R}^m$. However we do not need this formula in the sequel (compare [16]).

The pullback via Θ of the L^2 -scalar product of $H_0(I, \mathbb{R}^m)$ is a Riemannian structure g on $H_1^p(I, U)$ which induces for every compact neighborhood A of p in U a complete metric on $H_1^p(I, A) \subset H_1^p(I, U)$. If $\varphi \in H_1^p(I, U)$ and $X, Y \in H_1^p(\varphi)$, then $g_\varphi(X, Y) = \int_0^1 \langle d\Theta_\varphi X(t), d\Theta_\varphi Y(t) \rangle dt$.

The linear subspace $\{X \in H_1^p(\varphi) | X(1) = 0\} \subset H_1^p(\varphi)$ is closed in $H_1^p(\varphi)$; hence its g_φ -orthogonal complement $J(\varphi)$ is an m -dimensional linear subspace of $H_1^p(\varphi)$. We have

Lemma 2.5. $J(\varphi) = \{X \in H_1^p(\varphi) | \frac{d}{dt}(d\Theta_\varphi X)(t) - a(\varphi'(t), (d\Theta_\varphi X)(t)) \equiv 0\}$.

Proof. Let $Y \in H_1^p(\varphi)$ be the preimage under $d\Theta_\varphi$ of a curve of class H_1 in \mathbb{R}^m . By Lemma 2.3 for every $X \in H_1^p(\varphi)$ we have

$$g_\varphi(X, Y) = \langle \theta X(1), (d\Theta_\varphi Y)(1) \rangle$$

$$- \int_0^1 \left\langle \theta X(t), \frac{d}{dt}(d\Theta_\varphi Y)(t) - a(\varphi'(t), (d\Theta_\varphi Y)(t)) \right\rangle dt.$$

Thus any solution $c: I \rightarrow V$ of the differential equation

$$(1) \quad c'(t) = a(\varphi'(t))c(t)$$

is the image under $d\Theta_\varphi$ of an element of $J(\varphi)$. Now (1) is a linear differential equation whose coefficients (i.e., the entries of the matrix rep-

representing $a(\varphi'(t))$) are as regular in t as the map $t \rightarrow \theta\varphi'(t)$, i.e., they are square integrable. Thus (1) admits precisely m linear independent solutions which shows the lemma. q.e.d.

If φ has a continuous derivative, the existence of an m -dimensional space of solutions of (1) follows from the standard theory for solutions of ordinary differential equations with continuous coefficients. We include a proof for the general case since it provides us with norm estimates which are needed later.

For a curve φ of class H_1 in U and an element c of the Banach space $L^\infty(I, \mathbb{R}^m)$ of essentially bounded maps $I \rightarrow \mathbb{R}^m$ provided with the norm $|c| = \text{ess sup}_{t \in I} \|c(t)\|$, define $T_\varphi c(s) = \int_0^s a(\varphi'(t))c(t) dt$. Thus c is a solution of (1) with $c(0) = c_0$ for some $c_0 \in \mathbb{R}^m$ if and only if $c - T_\varphi c \equiv c_0$.

Let $\|L\|$ be the operator norm of a linear endomorphism L of the Euclidean space \mathbb{R}^m . Then $\varphi \in H_1^p(I, U)$ means $\nu(\varphi) = \int_0^1 \|a(\varphi'(t))\| dt < \infty$.

Lemma 2.6. *For every $\varphi \in H_1^p(I, U)$, $\text{Id} - T_\varphi$ is a continuous invertible linear automorphism of $L^\infty(I, \mathbb{R}^m)$. The operator norm of $(\text{Id} - T_\varphi c)^{-1}$ does not exceed $(2\nu(\varphi) + 2)^{2\nu(\varphi)+1}$.*

Proof. Let $c \in L^\infty(I, V)$; then $\|T_\varphi c(s)\| = \|\int_0^s a(\varphi'(t))c(t) dt\| \leq \nu(\varphi)|c|$, i.e., T_φ is a continuous linear endomorphism of $L^\infty(I, \mathbb{R}^m)$ whose operator norm does not exceed $\nu(\varphi)$.

Let $k > 0$ be the smallest integer which is not smaller than $2\nu(\varphi)$ and choose a partition $0 = s(0) < s(1) < \dots < s(k) = 1$ of I such that $\int_{s(j)}^{s(j+1)} \|a(\varphi'(t))\| dt \leq \frac{1}{2}$ for all $j < k$. Define $\psi_j(t) = s(j) + t(s(j+1) - s(j))$ and $\varphi_j(t) = \varphi(\psi_j(t))$, for $t \in I$, and let $c \in L^\infty(I, \mathbb{R}^m)$, $c_j(t) = c(\psi_j(t))$. Since the operator norm of T_{φ_j} is not larger than $\frac{1}{2}$, $\text{Id} - T_{\varphi_j}$ is invertible (see [15, p. 231]) and $(\text{Id} - T_{\varphi_j})^{-1} = \sum_{i=0}^{\infty} T_{\varphi_j}^i$, in particular the operator norm of $(\text{Id} - T_{\varphi_j})^{-1}$ does not exceed $\sum_{i=0}^{\infty} 2^{-i} = 2$. Hence there is a unique $\alpha \in L^\infty(I, \mathbb{R}^m)$ such that we have $(\text{Id} - T_{\varphi_j})\alpha_j = c_j + \int_0^{s(j)} a(\varphi'(t))\alpha(t) dt$ with $\alpha_j(t) = \alpha(\psi_j(t))$ ($j < k$). Then

$$\begin{aligned} c(\psi_j(t)) &= c_j(t) = \alpha_j(t) - \int_0^t a(\varphi'_j(s))\alpha_j(s) ds - \int_0^{s(j)} a(\varphi'(s))\alpha(s) ds \\ &= \alpha(\psi_j(t)) - \int_0^{\psi_j(t)} a(\varphi'(s))\alpha(s) ds, \end{aligned}$$

which means $(\text{Id} - T_\varphi)\alpha = c$. This shows that $\text{Id} - T_\varphi$ is invertible. Moreover we have

$$|\alpha_j| \leq 2 \left(|c_j| + \left\| \int_0^{s(j)} a(\varphi'(t))\alpha(t) dt \right\| \right) \leq 2 \left(|c| + \nu(\varphi) \sup_{i < j} |\alpha_i| \right),$$

and inductively $|\alpha| = \sup_{j < k} |\alpha_j| \leq 2^k (1 + \nu(\varphi))^k |c|$. This means that the operator norm of $(\text{Id} - T_\varphi)^{-1}$ does not exceed $2^k (1 + \nu(\varphi))^k$ which is the claim.

Remark 2.7. Let $S \subset U$ be a smooth k -dimensional submanifold with tangent bundle TS . Then $\{\varphi \in H_1^p(I, U) | \varphi(1) \in S\}$ is a smooth submanifold of $H_1^p(I, U)$ of codimension $m - k$. Its tangent space at φ consists of all $X \in H_1^p(\varphi)$ with $X(1) \in TS$. Lemma 2.5 thus shows that the g_φ -orthogonal complement of this tangent space is just the $(m - k)$ -dimensional vector space $\{X \in J(\varphi) | (d\Theta_\varphi X)(1) \in (TS)^\perp\}$.

Remark 2.8. If M is a Lie group with identity $e = p$, and the vector fields X^1, \dots, X^m are left-invariant, then the Lie algebra \mathfrak{M} of M can naturally be identified with \mathbb{R}^m . With this identification, θ is the canonical left-invariant 1-form on M with values in \mathfrak{M} (see [9]). Thus $2d\theta(X, Y) = (\text{ad } X)(Y)$, where as usual ad denotes the adjoint representation of \mathfrak{M} . Let Ad be the adjoint representation of M in \mathfrak{M} , and denote by Ad_u^* the adjoint of Ad_u for $u \in M$. If $\varphi \in H_1^e(I, M)$, then for every $c_0 \in \mathfrak{M}$ the curve $t \rightarrow \text{Ad}_{\varphi(t)}^* c_0$ satisfies the differential equation (1) of Lemma 2.5. Thus in this case $J(\varphi) = \{X \in H_1^e(\varphi) | (d\Theta_\varphi X)(t) = \text{Ad}_{\varphi(t)}^* c_0 \text{ for some } c_0 \in \mathfrak{M}\}$.

3. The manifold of curves tangent to Q

In this section we begin to investigate the submanifold HQ of $H_1^p(I, U)$ of curves which are tangent almost everywhere to Q .

Identify Q with the subspace $\theta(Q) \cong \mathbb{R}^q$ of \mathbb{R}^m . The set $HQ = \Theta^{-1}H_0(I, Q)$ of curves which are tangent almost everywhere to Q is a closed submanifold of $H_1^p(I, U)$. If $\tilde{P}: V \rightarrow Q$ denotes the $\langle \cdot, \cdot \rangle$ -orthogonal projection, then for every $\varphi \in HQ$ the g_φ -orthogonal projection P of $H_1^p(\varphi)$ onto the tangent space HQ_φ of HQ at φ is defined by $PX = (d\Theta_\varphi)^{-1} \tilde{P} d\Theta_\varphi X$.

Let $K(\varphi) \sim H_0(I, Q^\perp)$ be the kernel of the projection P , and define $\Omega(\varphi) = \{X \in HQ_\varphi \mid X(1) = 0\}$. Then $H_1^p(\varphi) = \Omega(\varphi) \oplus (J(\varphi) + K(\varphi))$, and the g_φ -orthogonal complement $\Omega(\varphi)^\perp$ of $\Omega(\varphi)$ in HQ_φ is contained in $P(J(\varphi) + K(\varphi)) = PJ(\varphi)$. Thus $\Omega(\varphi)^\perp = \{X \in HQ_\varphi \mid (d\Theta_\varphi X)(t) = \tilde{P}c(t) \text{ for } c \in H_1(I, V) \text{ with } c'(t) = a(\varphi'(t))c(t)\}$.

Let $R: HQ \rightarrow U, \varphi \rightarrow \varphi(1)$ be the endpoint map. Then the rank of R at φ equals the dimension of $\Omega(\varphi)^\perp$, and this dimension varies between $q = \dim Q$ at the constant curve $\varphi(I) = p$ and $m = \dim M$. In particular for $u \in U$ the closed subset $R^{-1}(u)$ of HQ may not be a submanifold.

However the set $\{\varphi \in HQ \mid \text{rank } R_\varphi = m\}$ is clearly open in HQ . If M is a Lie group, then Remark 2.8 shows that it is even open as a subset of HQ with the C^0 -topology. A similar property holds in general. For its formulation let dist again be the distance on M induced by the Riemannian metric, and recall that the space $C^0(I, M)$ of continuous curves in M with the distance $d_\infty(\varphi, \psi) = \sup\{\text{dist}(\varphi(t), \psi(t)) \mid t \in I\}$ is a Banach manifold, in particular a locally complete metric space. Let $E: HQ \rightarrow \mathbb{R}$ be the restriction to HQ of the energy function $\varphi \rightarrow \frac{1}{2} \int_0^1 \|\varphi'(t)\|^2 dt$. First we have

Lemma 3.1. *Let $\mu > 0$ and $\varphi \in HQ \cap E^{-1}[0, \mu]$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\sup_{t \in I} \|\int_0^t (\theta\gamma'(s) - \theta\varphi'(s)) ds\| < \varepsilon$ for all $\gamma \in HQ \cap E^{-1}[0, \mu]$ with $d_\infty(\gamma, \varphi) < \delta$.*

Proof. Let $U, X^1, \dots, X^m, \theta$ be as before and assume without loss of generality that there is a diffeomorphism Ψ of \mathbb{R}^m onto U with $\Psi(0) = p$. Define $c(t) = \Psi^{-1}\varphi(t)$ and denote by $\|L\|$ its operator norm for a linear map L between Euclidean vector spaces.

Let A be a compact neighborhood of $c(I)$ in \mathbb{R}^m and let

$$\rho = \sup\{\|d\Psi_u\|, \|d\Psi_{\Psi^{-1}u}^{-1}\| \mid u \in A\} < \infty.$$

By the smoothness of Ψ there is then $\sigma > 0$ such that for every $t \in I$ and every $u \in U$ with $\text{dist}(\varphi(t), u) < \sigma$

$$\|\theta(d\Psi)_{c(t)} - \theta(d\Psi)_{\Psi^{-1}u}\| < \varepsilon/8\rho\sqrt{\mu}.$$

Choose $n \geq 2$ such that $\text{dist}(\varphi(s), \varphi(t)) < \sigma$ for $|s - t| < 1/n$, and let $\delta < \sigma$ be sufficiently small that $d_\infty(\varphi, \gamma) < \delta$ implies $\|c(t) - \Psi^{-1}\gamma(t)\| < \varepsilon/16n\rho$ for all $t \in I$. Let $\gamma \in HQ \cap E^{-1}[0, \mu]$ with $d_\infty(\gamma, \varphi) < \delta$ and

define $\bar{c}(t) = \Psi^{-1}\gamma(t)$. Then

$$\begin{aligned}
 & \sup_{t \in I} \left\| \int_0^t (\theta\gamma'(s) - \theta\varphi'(s)) ds \right\| \\
 & \leq \sup_{t \in I} \left\| \int_0^t (\theta(d\Psi)_{\bar{c}(s)} - \theta(d\Psi)_{c(s)}) \bar{c}'(s) ds \right\| \\
 & \quad + \sup_{t \in I} \left\| \int_0^t \theta(d\Psi)_{c(s)} (\bar{c}'(s) - c'(s)) ds \right\| \\
 & \leq \int_0^1 \|(\theta(d\Psi)_{\bar{c}(s)} - \theta(d\Psi)_{c(s)}) \bar{c}'(s)\| ds \\
 & \quad + \sup_{k \leq n-1} \left\| \sum_{j=0}^k \theta(d\Psi)_{c(j/n)} \int_{j/n}^{(j+1)/n} (\bar{c}'(s) - c'(s)) ds \right\| \\
 & \quad + \sup_{k < n} \sup_{r < 1/n} \left\| \theta(d\Psi)_{c(k/n)} \int_{k/n}^{k/n+r} (\bar{c}'(s) - c'(s)) ds \right\| \\
 & \quad + \sum_{j \geq 0} \int_{j/n}^{(j+1)/n} \|(\theta(d\Psi)_{c(s)} - \theta(d\Psi)_{c(j/n)}) (\bar{c}'(s) - c'(s))\| ds \\
 & \leq \frac{\varepsilon \rho \sqrt{\mu}}{8} \int_0^1 \|\bar{c}'(s)\| ds \\
 & \quad + 2 \sum_{j=0}^n \rho \|\bar{c}(j/n) - c(j/n)\| \\
 & \quad + \rho \sup_{j \leq n} \sup_{r < 1/n} (\|\bar{c}(\tau + j/n) - c(\tau + j/n)\| + \|\bar{c}(j/n) - c(j/n)\|) \\
 & \quad + \frac{\varepsilon \rho \sqrt{\mu}}{8} \int_0^1 \|\bar{c}'(s) - c'(s)\| ds.
 \end{aligned}$$

Since $\int_0^1 \|c'(s)\| ds \leq \rho\sqrt{\mu}$ and $\int_0^1 \|\bar{c}'(s)\| ds \leq \rho\sqrt{\mu}$, the latter sum does not exceed ε which yields the claim. q.e.d.

Recall the definition of the automorphisms T_γ of $L^\infty(I, V)$ ($\gamma \in H_1^p(I, U)$) preceding Lemma 2.6. From Lemma 3.1 we obtain

Corollary 3.2. *Let $\mu > 0$ and $\varphi \in HQ \cap E^{-1}[0, \mu]$. Then for every $\varepsilon > 0$ there is $\delta > 0$ such that $\|(\text{Id} - T_\varphi)^{-1} - (\text{Id} - T_\gamma)^{-1}\| < \varepsilon$ for all $c \in \mathbb{R}^m$ with $\|c\| = 1$ and all $\gamma \in HQ \cap E^{-1}[0, \mu]$ with $d_\infty(\gamma, \varphi) < \delta$.*

Proof. Choose a compact neighborhood B of $\varphi(I)$ in \bar{U} . Since a is a smooth 1-form on U with values in the linear space of endomorphisms of \mathbb{R}^m , there is $\alpha > 0$ such that $\|a(X)\| \leq \alpha\|X\|$ for all $u \in B$ and

$X \in T_u M$. This means $\nu(\gamma) \leq \alpha\sqrt{\mu}$ for all $\gamma \in HQ \cap E^{-1}[0, \mu) \cap H_1^p(I, B) = \tilde{H}$.

Let $\varepsilon > 0$ and choose $\bar{\varepsilon} < \varepsilon/2(2\sqrt{\mu} + 2)^{2\sqrt{\mu}+1}$. Let Σ be the unit sphere in \mathbb{R}^m . Then the map $\psi: \Sigma \times I \rightarrow V$, $(c, t) \rightarrow ((\text{Id} - T_\varphi)^{-1}c)(t)$ is continuous. Hence there is $\rho > 0$ such that $\|\psi_c(t) - \psi_c(s)\| < \bar{\varepsilon}/(\alpha\sqrt{\mu} + 1)$ for all $c \in \Sigma$ and $s, t \in I$ with $|s - t| < \rho$. Moreover

$$\sigma = \sup\{\|\psi_c(t)\| \mid c \in \Sigma, t \in I\} < \infty.$$

Let $k > 1/\rho$ and define $\tilde{\psi}_c(t) = \psi_c([kt]/k)$ for $c \in \Sigma$. Then $\tilde{\psi}_c \in L^\infty(I, \mathbb{R}^m)$ and $|\tilde{\psi}_c - \psi_c| < \bar{\varepsilon}/(\alpha\sqrt{\mu} + 1)$, where $|\cdot|$ is the norm in $L^\infty(I, \mathbb{R}^m)$ as before. Let $\gamma \in \tilde{H}$; since the operator norm of $\text{Id} - T_\gamma$ does not exceed $\alpha\sqrt{\mu} + 1$, we have $|(\text{Id} - T_\gamma)(\psi_c - \tilde{\psi}_c)| < \bar{\varepsilon}$.

For $\gamma \in \tilde{H}$ and $t \in I$, define a linear endomorphism $A_\gamma(t)$ of \mathbb{R}^m by $A_\gamma(t) = \int_0^t a(\gamma'(s)) ds$ which means $\lambda A_\gamma(t) = \int_0^t \lambda(a(\gamma'(s))) ds$ for every linear functional λ on the vector space of linear endomorphisms of \mathbb{R}^m . Then

$$\begin{aligned} T_\gamma \tilde{\psi}_c(t) &= \int_{[tk]/k}^t a(\gamma'(t)) \psi_c([tk]/k) dt \\ &\quad + \sum_{j=0}^{[tk]-1} \int_{j/k}^{(j+1)/k} a(\gamma'(t)) \psi_c(j/k) dt \\ &= (A_\gamma(t) - A_\gamma([tk]/k)) \psi_c([tk]/k) \\ &\quad + \sum_{j=0}^{[tk]-1} (A_\gamma(j+1)/k - A_\gamma(j/k)) \psi_c(j/k). \end{aligned}$$

Since $a(X_u, \theta Y_u) \in \mathbb{R}^m$ depends smoothly on $u \in U$ for smooth vectors fields X, Y on U , Lemma 3.1 shows that there is $\delta > 0$ such that for all $\gamma \in HQ \cap E^{-1}[0, \mu)$ with $d_\infty(\varphi, \gamma) < \delta$ we have $\gamma \in \tilde{H}$ and

$$\sup_{t \in I} \|A_\gamma(t) - A_\varphi(t)\| < \bar{\varepsilon}/2(k + 1)\sigma.$$

By the definition of σ this means

$$|T_\varphi \tilde{\psi}_c - T_\gamma \tilde{\psi}_c| \leq 2 \sum_{j=0}^k \|A_\varphi(j/k) - A_\gamma(j/k)\| \sigma < \bar{\varepsilon},$$

hence

$$\begin{aligned} |(\text{Id} - T_\gamma)\tilde{\psi}_c - c| &\leq |(\text{Id} - T_\gamma)\tilde{\psi}_c - (\text{Id} - T_\varphi)\tilde{\psi}_c| \\ &\quad + |(\text{Id} - T_\varphi)\tilde{\psi}_c - (\text{Id} - T_\varphi)\psi_c| < 2\bar{\varepsilon}. \end{aligned}$$

Now by Lemma 2.6 the operator norm of $(\text{Id} - T_\gamma)^{-1}$ does not exceed $(2\alpha\sqrt{\mu} + 2)2^{2\sqrt{\mu}+1}$; from this we obtain

$$|(\text{Id} - T_\gamma)^{-1}c - \psi_c| \leq |(\text{Id} - T_\gamma)^{-1}c - \tilde{\psi}_c| + |\tilde{\psi}_c - \psi_c| < \varepsilon$$

which is the claim.

Corollary 3.3. *For every $\mu > 0$ and every $k \leq m$ the set $\{\varphi \in HQ \mid \text{rank } R_\varphi \geq k\} \cap E^{-1}[0, \mu)$ is open in $HQ \cap E^{-1}[0, \mu) \subset (C^0(I, U), d_\infty)$.*

Proof. Let $\varphi \in HQ$ with $E(\varphi) < \mu$ and $\text{rank } R_\varphi = k$. Let $\Sigma \subset \mathbb{R}^m$ be the unit sphere in the orthogonal complement of the intersection of V with $(\text{Id} - T_\varphi)L^\infty(I, Q^\perp)$. Then there is by Lemma 2.5 a number $\varepsilon > 0$ such that $\sup_{t \in I} \|\tilde{P}(\text{Id} - T_\varphi)^{-1}c_0\| > 2\varepsilon$ for all $c_0 \in \Sigma$. By Lemma 3.2 we can find $\delta > 0$ such that $|(\text{Id} - T_\gamma)^{-1}c_0 - (\text{Id} - T_\varphi)^{-1}c_0| < \varepsilon$ for all $c_0 \in \Sigma$ and all $\gamma \in HQ \cap E^{-1}[0, \mu)$ with $d_\infty(\varphi, \gamma) < \delta$. For such a γ we have $\sup_{t \in I} \|\tilde{P}(\text{Id} - T_\gamma)^{-1}c_0\| > \varepsilon$ for all $c_0 \in \Sigma$, which means by Lemma 2.5 that $\text{rank } R_\gamma = m$.

4. Critical points of the energy function

Let $\psi: (-\varepsilon, \varepsilon) \rightarrow HQ$ be a variation of $\varphi = \psi_0$ with variation vector field $X = \frac{\partial}{\partial s}\psi|_{s=0}$. For the derivative at $s = 0$ of the energy function E on HQ we obtain

$$\frac{\partial}{\partial s} E(\psi_s)|_{s=0} = \int_0^1 \langle (d\Theta_\varphi X)(t), \theta\varphi'(t) \rangle dt,$$

i.e., $(d\Theta_\varphi)^{-1}\theta\varphi'$ is the gradient of E at φ .

Call $\varphi \in HQ$ a *critical point* of E if $(d\Theta_\varphi)^{-1}\theta\varphi' \in \Omega(\varphi)^\perp$. If the rank of R at φ is maximal, then there is a neighborhood A of φ in HQ such that $R^{-1}(\varphi(1)) \cap A$ is a smooth submanifold of A , and φ is thus critical for the restriction of E to this submanifold in the usual sense.

This immediately shows that every minimizing d_c -geodesic φ with $\text{rank } R_\varphi = m$ is necessarily a critical point for E .

Define a smooth $(2, 1)$ -tensor field \bar{a} on U by $\theta\bar{a}_u(X, Y) = a(X)\theta Y$ ($u \in U, X, Y \in T_uM$).

Lemma 4.1. *The critical points of E are smooth curves parametrized proportional to arc length.*

Proof. If $\varphi \in HQ$ is a critical point of E , then $(d\Theta_\varphi)^{-1}\theta\varphi'$ is the projection in HQ_φ of an element of $J(\varphi)$. Thus by Lemma 2.5 there is

a function $\alpha: I \rightarrow Q^\perp \subset \mathbb{R}^m$ such that

$$(2) \quad \frac{d}{dt}\theta\phi'(t) + \frac{d}{dt}\alpha(t) - a(\phi'(t), \theta\phi'(t) + \alpha(t)) \equiv 0.$$

If, by abuse of notation, we denote by \tilde{P} the $\langle \cdot, \cdot \rangle$ -orthogonal projection of TM onto Q , then (2) transforms to

$$(2') \quad \frac{d}{dt}\theta c(t) = \theta\bar{a}(\tilde{P}c(t), c(t))$$

which is a system of first order differential equations on TU with C^∞ -coefficients. Thus every solution of (2') is smooth, and moreover (2) shows $\langle \frac{d}{dt}\theta\phi'(t), \theta\phi'(t) \rangle \equiv 0$, i.e., critical points of E are parametrized proportional to arc length. q.e.d.

Now for every initial condition $X \in T_pM$ there is a unique maximal solution $\tilde{\lambda}(X)$ of (2') which depends smoothly on X . $\tilde{\lambda}(X)$ projects onto a smooth curve $\lambda(X)$ in U which is tangent to Q and parametrized proportional to arc length. Since, by definition, U is just the open d_c -ball of radius ρ around $p = \lambda(X)(0)$, for every $X \in T_pM$ with $\|\tilde{P}X\| < \rho$ the curve $\lambda(X)$ is defined on I and $\lambda(X)|_I \in HQ$ is a critical point of E . Hence $X \rightarrow \lambda(X)(1)$ defines a smooth map \exp_p^c of $\tilde{U} = \{X \in T_pM \mid \|\tilde{P}X\| < \rho\}$ into U . Now $\theta\frac{d}{dt}\lambda(X)(bt) = b\theta\lambda(X)'(bt)$ shows $\theta\tilde{\lambda}(bX)(t) = b\tilde{\lambda}(X)(bt)$ or $\lambda(bX)(t) = \lambda(X)(bt)$ for all $b \in \mathbb{R}$. Thus for every $X \in \tilde{U}$ we have $\exp_p^c(tX) = \lambda(X)(t)$ ($t \in I$), i.e., $t \rightarrow \exp_p^c(tX)$ ($t \in I$) is a critical point of E .

Lemma 4.2. *Let $X \in T_pM$ be such that $\lambda(X)$ is defined on I and $\text{rank } R_\lambda(X) = m$. Then for every $\mu > \|\tilde{P}X\|$ there is $\delta > 0$ and $\beta < \infty$ such that $d_\infty(\lambda(X), \lambda(Y)) < \delta$ and $\|\tilde{P}Y\| < \mu$ implies $\|Y\| < \beta$.*

Proof. Let Σ be the unit sphere in \mathbb{R}^m . Since $\text{rank } R_{\lambda(X)} = m$, there is then $\alpha > 0$ such that $|\tilde{P}(\text{Id} - T_{\lambda(X)})^{-1}c| > \alpha|(\text{Id} - T_{\lambda(X)})^{-1}c|$ for all $c \in \Sigma$.

Let $\sigma_1 = \inf\{|(\text{Id} - T_{\lambda(X)})^{-1}c| \mid c \in \Sigma\}$ and $\sigma_2 = \sup\{|(\text{Id} - T_{\lambda(X)})^{-1}c| \mid c \in \Sigma\}$. By Corollary 3.2 we can choose $\delta > 0$ such that

$$\sup_{c \in \Sigma} |((\text{Id} - T_{\lambda(X)})^{-1} - (\text{Id} - T_\gamma)^{-1})c| < \alpha\sigma_1/2$$

for all $\gamma \in HQ \cap E^{-1}[0, \mu^2]$ with $d_\infty(\lambda(X), \gamma) < \delta$. For such a γ we then have

$$|\tilde{P}(\text{Id} - T_\gamma)^{-1}c| > \alpha\sigma_1/2 > \alpha\sigma_1\sigma_2|(\text{Id} - T_\gamma)^{-1}c|/4$$

for all $c \in \Sigma$. Hence if $Y \in T_p M$ with $\|\tilde{P}Y\| < \mu$ and $d_\infty(\lambda(X), \lambda(Y)) < \delta$, then $E(\lambda(Y)) < \mu^2$ and

$$\begin{aligned} |\tilde{P}(\text{Id} - T_{\lambda(Y)})^{-1}\theta Y| &= \|\tilde{P}Y\| > \alpha\sigma_1\sigma_2|(\text{Id} - T_{\lambda(Y)})^{-1}\theta Y|/4 \\ &\geq \alpha\sigma_1\sigma_2\|Y\|/4. \end{aligned}$$

This yields the claim. q.e.d.

Recall that a d_c -geodesic is a curve φ in M , which is parametrized proportional to arc length and realizes locally the d_c -distance of its curve points. If the closure of U in M is compact (which is always true if we choose U small enough), then every $u \in U$ can be joined to p by a minimizing d_c -geodesic (see [6, 17]).

Any such geodesic which is parametrized on I is necessarily a critical point of E . This was stated in [17], however the proof provided there is only valid in the strong bracket generating case (where it also follows from the fact that the map R is of maximal rank on each nontrivial curve in $H_1^p(I, U)$). The general case was established by Bär [1]. In particular the map $\exp_p^c: \tilde{U} \rightarrow U$ is surjective.

Corollary 4.3. *Let $u \in U$ be a regular value of \exp_p^c . Then the set $(\exp_p^c)^{-1}(u) \cap \{Y \mid \|\tilde{P}Y\| = d_c(p, u)\}$ is finite.*

Proof. The set $A = (\exp_p^c)^{-1}(u) \cap \{Y \mid \|\tilde{P}Y\| = d_c(p, u)\}$ is nonempty and closed in $T_p M$. Since u is a regular value for \exp_p^c , A is moreover discrete. Assume that there is a sequence $\{X_k\} \subset A$ such that $\|X_k\| \rightarrow \infty$ ($k \rightarrow \infty$). Then $\lambda(X_k)$ is a minimizing geodesic joining p to u ; its energy equals $d_c(p, u)^2$. Thus by passing to a subsequence we may assume that the curves $\lambda(X_k)$ converges in $(C^0(I, U), d_\infty)$ to a curve φ which is necessarily a minimizing d_c -geodesic joining p to u . Then $\varphi = \lambda(X)$ for some $X \in A$ and Lemma 4.2 shows that there is $k_0 > 0$ and $\beta < \infty$ such that $\|X_k\| < \beta$ for all $k > k_0$. This contradicts the assumption that $\|X_k\| \rightarrow \infty$ and shows that A is bounded, hence finite.

5. Calculus of variation

Since the rank of \exp_p^c at $0 \in T_p M$ is not maximal the above considerations do not necessarily imply that $\lambda(X)$ is a d_c -geodesic for every $X \in \tilde{U}$, i.e., is locally minimizing with respect to d_c . To show that this is nevertheless true we compute the variation of the energy at the critical point $\lambda(X)$ ($X \in \tilde{U}$).

Lemma 5.1 (*First variational formula*). Let $X \in T_p M$ be such that $\lambda(X)$ is defined on I and let $\psi: (-\varepsilon, \varepsilon) \rightarrow HQ$ be a variation of $\lambda(X) = \psi_0$ with variation vector field $Y = \frac{\partial}{\partial s} \psi|_{s=0}$. Then $\frac{d}{ds} E(\psi_s)|_{s=0} = \langle Y(1), \tilde{\lambda}(X)(1) \rangle$.

Proof. With $\varphi = \lambda(X)$ we have

$$\begin{aligned} \frac{d}{ds} E(\psi_s)|_{s=0} &= \int_0^1 \langle (d\Theta_\varphi Y)(t), \theta\varphi'(t) \rangle dt \\ &= \int_0^1 \langle (d\Theta_\varphi Y)(t), \theta\tilde{\lambda}(X)(t) \rangle dt \\ &= \int_0^1 \left\langle \frac{d}{dt} \theta Y(t), \theta\tilde{\lambda}(X)(t) \right\rangle dt \\ &\quad + \int_0^1 \langle \theta Y(t), a(\varphi'(t), \theta\tilde{\lambda}(X)(t)) \rangle dt \\ &= \langle Y(1), \tilde{\lambda}(X)(1) \rangle. \quad \text{q.e.d.} \end{aligned}$$

Recall that $d\theta$ is a smooth 2-form on U with values in \mathbb{R}^m . Hence for $u \in U$ and every tangent vector $Y \in T_u M$ the derivative $Y(d\theta)$ of $d\theta$ in the direction of Y is a bilinear mapping of $T_u M$ into \mathbb{R}^m depending smoothly on Y .

It will be convenient, furthermore, to use the following notational convention: Recall that for every $u \in U$ the restriction of θ to $T_u M$ is a linear isomorphism of $T_u M$ onto \mathbb{R}^m , i.e., for every $W \in \mathbb{R}^m$ there is a unique $W(u) \in T_u M$ such that $\theta W(u) = W$. Thus whenever no confusion about the base point u is possible we can write $d\theta(W, Z)$ or $d\theta(W, Z(u))$ or $d\theta(W(u), Z)$ to denote the vector $d\theta(W(u), Z(u)) \in \mathbb{R}^m$. Similarly we denote by $a(W)$ the linear map $a(W(u))$ ($u \in U, W, Z \in \mathbb{R}^m$). With this convention the second variational formula for E can be expressed as follows:

Lemma 5.2 (*Second variational formula*). Let $X \in T_p M$ be such that $\lambda(X)$ is defined on I , and let $\psi: (-\varepsilon, \varepsilon)^2 \rightarrow HQ$ be a 2-parameter variation of $\varphi = \psi(0, 0)$ with fixed endpoints $\psi(-\varepsilon, \varepsilon)^2(1) = \varphi(1)$ and variation vector fields $Y = \frac{\partial}{\partial s} \psi|_{u=s=0}$, $Z = \frac{\partial}{\partial u} \psi|_{u=s=0}$. Then

$$\begin{aligned} \frac{\partial}{\partial u} \frac{\partial}{\partial s} E(\psi(u, s))|_{u=s=0} &= \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt \\ &\quad + \int_0^1 2 \langle (Z(t) d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t) \rangle dt. \end{aligned}$$

Proof. Since

$$\begin{aligned} \frac{\partial}{\partial u} \frac{\partial}{\partial s} E(\psi(u, s))|_{u=s=0} &= \int_0^1 \left\langle \frac{\partial}{\partial u} \frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \psi, \theta \frac{\partial}{\partial t} \psi \right\rangle_{u=s=0} dt \\ &+ \int_0^1 \langle (d\Theta_\varphi Y)(t), (d\Theta_\varphi Z(t)) \rangle dt, \end{aligned}$$

we have to transform the first integral. Define $W(t) = \frac{\partial}{\partial u} \theta \frac{\partial}{\partial s} \Psi|_{u=s=0}$; then

$$\frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \Psi = \frac{\partial}{\partial t} \theta \frac{\partial}{\partial s} \Psi + 2d\theta \left(\frac{\partial}{\partial t} \Psi, \frac{\partial}{\partial s} \Psi \right)$$

yields

$$\begin{aligned} \frac{\partial}{\partial u} \frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \Psi|_{u=s=0} &= \frac{d}{dt} W(t) + 2d\theta(\varphi'(t), W(t)) + 2d\theta(d\Theta_\varphi Z(t), Y(t)) \\ &+ 2(Z(t)d\theta)(\varphi'(t), Y(t)). \end{aligned}$$

Since $W(1) = 0$ we have $\int_0^1 \langle (d\Theta_\varphi W)(t), \theta \tilde{\lambda}(X)(t) \rangle dt = 0$, i.e.,

$$\begin{aligned} &\int_0^1 \langle (d\Theta_\varphi W)(t), \theta \varphi'(t) \rangle dt \\ &= - \int_0^1 \langle (d\Theta_\varphi W)(t), (\text{Id} - \tilde{P})\theta \tilde{\lambda}(X)(t) \rangle dt. \end{aligned}$$

But $\frac{\partial}{\partial u} \frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \psi \subset Q = \tilde{P}\mathbb{R}^m$ shows

$$\begin{aligned} &(\tilde{P} - \text{Id})(d\Theta_\varphi W)(t) \\ &= (\text{Id} - \tilde{P})(2d\theta(d\Theta_\varphi Z(t), Y(t)) + 2(Z(t)d\theta)(\varphi'(t), Y(t))); \end{aligned}$$

hence

$$\begin{aligned} &\int_0^1 \left\langle \frac{\partial}{\partial u} \frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \Psi|_{u=s=0}, \theta \varphi'(t) \right\rangle dt \\ &= \int_0^1 \langle 2d\theta(d\Theta_\varphi Z(t), Y(t)) + 2(Z(t)d\theta)(\varphi'(t), Y(t)), \theta \tilde{\lambda}(X)(t) \rangle dt, \end{aligned}$$

and from this the lemma follows.

Remark 5.3. Assume again that M is a Lie group and that the vector fields X^1, \dots, X^m are left-invariant. By Remark 2.8 we have $J(\varphi) = \{X \in H_1^e(\varphi) | (d\Theta_\varphi X)(t) = \text{Ad}_{\varphi(t)}^* c_0 \text{ for some } c_0 \in \mathfrak{M}\}$. For $Y \in \mathfrak{M}$ let $(\text{ad}(Y))^*$ be the adjoint of the linear endomorphism $\text{ad}(Y)$ of \mathfrak{M} . Then the formula of Lemma 5.2 reduces to

$$\begin{aligned} &\frac{\partial}{\partial u} \frac{\partial}{\partial s} E(\Psi(u, s))|_{u=s=0} \\ &= \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - (\text{ad}(\theta Y(t)))^*(\theta \tilde{\lambda}(X)(t)) \rangle dt. \end{aligned}$$

Thus if Y is contained in the zero space of the Hessian of E at φ , then Y is a solution of the differential equation

$$(3) \quad (d\Theta_\varphi Y)(t) = \tilde{P}(\text{Ad}_{\varphi(t)}^* c_0 + (\text{ad}(\theta Y(t)))^*(\theta\tilde{\lambda}(X)(t)))$$

for some $c_0 \in \mathfrak{M}$. Every solution of (3) is uniquely determined by the choice of c_0 and the initial condition $Y(0)$; in particular the dimension of the vector space of solutions of (3) vanishing at $t = 0$ equals $\text{rank } R_\varphi = \dim \Omega(\varphi)^\perp$ which in contrast to the fact that the Riemannian situation may be strictly smaller than $\dim M$.

Next we want to compute the zero space of the Hessian of E . For this the following notation will be useful: Given $\varphi \in HQ$ and $Z \in H_0(I, \mathbb{R}^m)$ there is a unique vector field $fZ \in H_1^p(\varphi)$ such that $Z(t) = \frac{d}{dt}\theta(fZ)(t) - a(\varphi'(t), \theta(fZ)(t))$. Write also $(fZ)(t) = \int_0^t Z(\tau) d\tau$. For every $W \in H_1^p(\varphi)$ we then have

$$\frac{d}{dt}\langle W(t), (fZ)(t) \rangle = \langle d\Theta_\varphi W(t), \theta(fZ)(t) \rangle + \langle \theta W(t), Z(t) \rangle.$$

Now if ω is a $(2, 0)$ -tensor on U with values in \mathbb{R}^m , then for each $u \in U$ and $X \in T_uM$ the assignment $Y \rightarrow \omega(X, Y)$ is a linear map of T_uM into \mathbb{R}^m . We denote by $(\omega(X))^*$ its adjoint. With these notation we obtain

Corollary 5.4. *Under the assumptions of Lemma 5.2 we have*

$$\begin{aligned} & \frac{\partial}{\partial u} \frac{\partial}{\partial s} E(\psi(u, s))|_{u=s=0} \\ &= \int_0^1 \left\langle d\Theta_\varphi Z(t), d\Theta_\varphi Y(t) - \theta \int_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X)(\tau)) d\tau \right. \\ & \quad \left. - \theta \int_0^t 2((Y(\tau) d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau) d\tau \right\rangle dt. \end{aligned}$$

Proof. The claim follows from Lemma 5.2 and the following computation:

$$\begin{aligned} & - \int_0^1 \langle d\Theta_\varphi Y(t), a(Z(t), \theta\tilde{\lambda}(X)(t)) \rangle dt \\ &= \int_0^1 \langle \theta Z(t), a(d\Theta_\varphi Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt, \end{aligned}$$

and since $Z(1) = 0$, integration by parts shows that the latter integral equals

$$- \int_0^1 \left\langle d\Theta_\varphi Z(t), \theta \int_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X)(\tau)) d\tau \right\rangle dt.$$

Analogously

$$\begin{aligned} & \int_0^1 \langle (Y(t) d\theta)(\varphi'(t), Z(t)), \theta\tilde{\lambda}(X)(t) \rangle dt \\ &= \int_0^1 \langle \theta Z(t), ((Y(t) d\theta)(\varphi'(t)))^* \theta\tilde{\lambda}(X)(t) \rangle dt \\ &= - \int_0^1 \left\langle d_{\Theta_\varphi} Z(t), \theta \int_0^t ((Y(\tau) d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau) d\tau \right\rangle dt. \quad \text{q.e.d.} \end{aligned}$$

Corollary 5.4 shows that if a field $Y \in HQ_\varphi$ is contained in the zero space of the Hessian of E at φ , then there is $\tilde{Y} \in J(\varphi)$ such that

$$(4) \quad \begin{aligned} d_{\Theta_\varphi} Y(t) = \tilde{P} \left(d_{\Theta_\varphi} \tilde{Y}(t) + \theta \int_0^t a(d_{\Theta_\varphi} Y(\tau), \theta\tilde{\lambda}(X)(\tau)) d\tau \right. \\ \left. + \theta \int_0^t 2((Y(\tau) d\theta)(\varphi'(\tau)))^* \theta\tilde{\lambda}(X)(\tau) d\tau \right). \end{aligned}$$

This differential equation can be transformed to a differential equation of the form $c'(t) = f(t, c(t))$ for some smooth function $f: I \times V \rightarrow V$ which is linear in the second variable as follows (then $c(t)$ is interpreted as $\theta Y(t)$): We have

$$\begin{aligned} & a(d_{\Theta_\varphi} Y(t), \theta\tilde{\lambda}(X)(t)) \\ &= \frac{d}{dt} a(Y(t), \theta\tilde{\lambda}(X)(t)) - (\varphi'(t)a)(Y(t), \theta\tilde{\lambda}(X)(t)) \\ &\quad - a(Y(t), a(\varphi'(t), \theta\tilde{\lambda}(X)(t))) \\ &\quad + 2a(d\theta(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t)), \end{aligned}$$

and hence

$$\begin{aligned} & \theta \int_0^t a(d_{\Theta_\varphi} Y(\tau), \theta\tilde{\lambda}(X)(\tau)) d\tau \\ &= a(Y(t), \theta\tilde{\lambda}(X)(t)) + \theta \int_0^t \left(a(\varphi'(\tau), a(Y(\tau), \theta\tilde{\lambda}(X)(\tau))) \right. \\ &\quad \left. - (\varphi'(\tau)a)(Y(\tau), \theta\tilde{\lambda}(X)(\tau)) \right. \\ &\quad \left. - a(Y(\tau), a(\varphi'(\tau), \theta\tilde{\lambda}(X)(\tau))) \right. \\ &\quad \left. + 2a(d\theta(\varphi'(\tau), Y(\tau)), \theta\tilde{\lambda}(X)(\tau)) \right) d\tau \\ &= \tilde{f}(t, \theta Y(t)), \end{aligned}$$

where $\tilde{f}: I \times V \rightarrow V$ is clearly linear in the second variable. Thus

$$(4') \quad \begin{aligned} \frac{d}{dt} \theta Y(t) = & \tilde{P} \left(d\Theta_\varphi \tilde{Y}(t) + \tilde{f}(t, \theta Y(t)) \right. \\ & \left. + \theta \int_0^t ((Y(\tau) d\theta)(\varphi'(\tau)))^* \theta \tilde{\lambda}(X)(\tau) d\tau \right) \\ & - 2d\theta(\varphi'(t), Y(t)) \end{aligned}$$

is clearly an equation of the required form.

Thus for every $\tilde{Y} \in J(\varphi)$ and every $Y_0 \in T_p M$ there is a unique solution Y of (4) with initial condition $Y(0) = Y_0$. Such a field is called a *Jacobi field* along φ .

By the linearity of (4') the Jacobi fields along φ form a vector space of dimension $m + \text{rank } R_\varphi$, and the zero space of the Hessian of E at φ consists exactly of the space of Jacobi fields along φ vanishing at $t = 0$ and $t = 1$.

As in the Riemannian situation the space of Jacobi fields vanishing at $t = 0$ equals the space of variational vector fields along φ of variations by geodesics.

Lemma 5.5. *Let $\psi(s, t) = \lambda(X_1 + sX_2)(t)$ for some $X_1, X_2 \in V$. Then $\frac{\partial}{\partial s} \psi|_{s=0}$ is the Jacobi field Y along $\varphi = \psi_0$ with initial condition $Y(0) = 0$ which is determined by the field $\tilde{Y} \in J(\varphi)$ with $d\Theta_\varphi \tilde{Y}(0) = X_2$.*

Proof. Let $\alpha(s, t) = (1 - \tilde{P})\theta \tilde{\lambda}(X_1 + sX_2)(t)$ and define $Y(t) = \frac{\partial}{\partial s} \psi(s, t)|_{s=0}$. Since

$$\frac{\partial}{\partial t} \theta \frac{\partial}{\partial t} \psi + \frac{\partial}{\partial t} \alpha - a \left(\frac{\partial}{\partial t} \psi, \theta \frac{\partial}{\partial t} \psi + \alpha \right) \equiv 0$$

and

$$\frac{\partial}{\partial s} \theta \frac{\partial}{\partial t} \psi|_{s=0} = d\Theta_\varphi Y,$$

we have

$$\begin{aligned} \left(\frac{\partial}{\partial s} \frac{\partial}{\partial t} \theta \frac{\partial}{\partial t} \psi + \frac{\partial}{\partial s} \frac{\partial}{\partial t} \alpha \right)_{s=0} &= \frac{d}{dt} d\Theta_\varphi Y + \frac{\partial}{\partial t} \frac{\partial}{\partial s} \alpha|_{s=0} \\ &= (Y(t)a)(\varphi'(t), \theta \varphi'(t) + \alpha(0, t)) \\ &\quad + a(d\Theta_\varphi Y(t), \theta \varphi'(t) + \alpha(0, t)) \\ &\quad + a \left(\varphi'(t), d\Theta_\varphi Y(t) + \frac{\partial}{\partial s} \alpha|_{s=0} \right), \end{aligned}$$

which means

$$\begin{aligned} & \frac{d}{dt} \left(d\Theta_\varphi Y(t) + \frac{\partial}{\partial s} \alpha|_{s=0} \right) - a \left(\varphi'(t), d\Theta_\varphi Y(t) + \frac{\partial}{\partial s} \alpha|_{s=0} \right) \\ &= (Y(t)a)(\varphi'(t), \theta\tilde{\lambda}(X_1)(t)) + a(d\Theta_\varphi Y(t), \theta\tilde{\lambda}(X_1)(t)). \end{aligned}$$

Let $\tilde{Y} \in J(\varphi)$ be such that $d\Theta_\varphi \tilde{Y}(0) = X_2 = d\Theta_\varphi Y(0) + \frac{\partial}{\partial s} \alpha(0, 0)$. Since by Lemma 2.5 $\frac{d}{dt} d\Theta_\varphi \tilde{Y}(t) - a(\varphi'(t), d\Theta_\varphi \tilde{Y}(t)) = 0$, it follows from the above equation that

$$\begin{aligned} d\Theta_\varphi Y(t) + \frac{\partial}{\partial s} \alpha &= \theta \int_0^t ((Y(\tau)a)(\varphi'(\tau), \theta\tilde{\lambda}(X_1)(\tau))) d\tau \\ &+ \theta \int_0^t a(d\Theta_\varphi Y(\tau), \theta\tilde{\lambda}(X_1)(\tau)) d\tau + d\Theta_\varphi \tilde{Y}(t), \end{aligned}$$

hence we only have to show that $(Y(t)a)(Z, W) = 2((Y(t) d\theta)Z)^* W$ for all $Z, W \in \mathbb{R}^m$. Let $X \in \mathbb{R}^m$. Then

$$\begin{aligned} \frac{\partial}{\partial s} \langle a_{\psi(s,t)}(Z, W), X \rangle_{s=0} &= \langle (Y(t)a)(Z, W), X \rangle_{s=0} \\ &= 2 \frac{\partial}{\partial s} \langle W, d\theta_{\psi(s,t)}(Z, X) \rangle_{s=0} \\ &= 2 \langle W, (Y(t)d\theta)(Z, X) \rangle \\ &= 2 \langle ((Y(t) d\theta)Z)^* W, X \rangle, \end{aligned}$$

which implies $(Y(t)a)(Z, W) = 2((Y(t) d\theta)Z)^* W$ as required.

Remark 5.6. (a) For $X \in T_p M$ let $\text{null}(X)$ be the dimension of the vector space of Jacobi fields along $\lambda(X)$ vanishing at $\lambda(X)(0)$ and $\lambda(X)(1)$. It then follows from Corollary 5.4 and Lemma 5.5 that the rank of exp_p^c at X equals $\text{rank } R_{\lambda(X)} - \text{null}(X)$. In particular if $\tilde{X} \in T_{\lambda(X)(1)} M$ is such that $\lambda(\tilde{X})(t) = \lambda(X)(1-t)$, then the rank of $\text{exp}_{\lambda(X)(1)}^c$ at \tilde{X} equals the rank of exp_p^c at X .

(b) By Sard's theorem almost every $u \in U$ is a regular value for exp_p^c . If $u \in U$ is such a point, then (a) shows that p is a regular value for exp_u^c . Let $X \in T_p M$ be such that $t \rightarrow \text{exp}_p^c tX$ is a minimizing geodesic joining p to u . Then exp_p^c has maximal rank at X and hence by (a) and 5.4 and 5.5 the zero space of the Hessian of E at $\lambda(X)$ vanishes. Since $\lambda(X)$ is minimizing, this means that the Hessian of E at $\lambda(X)$ is positive definite.

We now have a closer look at the Hessian of E at the critical point φ . Assume that $\rho \leq 2$ as in the beginning of this section is sufficiently small such that the closed d_c -ball of radius ρ around p is compact. For

$X \in T_p M$ with $\|\tilde{P}X\| < \rho$, $\varphi = \lambda(X)$ is defined on I and thus can be viewed as an element of HQ . For $Y, Z \in HQ_\varphi$ define

$$I_X(Y, Z) = \int_0^1 \langle (d\Theta_\varphi Z)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt + \int_0^1 2\langle (Z(t) d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t) \rangle dt.$$

Then we have

Lemma 5.7. *There is $\kappa \in (0, \rho/2]$ such that $I_X(Y, Y) > 0$ for all $X \in T_p M$ with $\|X\| < \kappa$ and all $0 \neq Y \in HQ_{\lambda(X)}$.*

Proof. Since $\lambda(sX)(t) = \lambda(X)(st)$ for all $X \in T_p M$ and $s, t \in I$, it suffices to show that there is $\kappa \in (0, \rho/2]$ such that for all $X \in T_p M$ with $\|\tilde{P}X\| = \rho/2$, all $\delta < \kappa/\|X\|$, and all $Y \in HQ_{\lambda(X)}$ which do not vanish identically on $[0, \delta]$ we have

$$I_\delta(Y, Y) = \int_0^\delta \langle (d\Theta_\varphi Y)(t), (d\Theta_\varphi Y)(t) - a(Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt + \int_0^\delta 2\langle (Y(t) d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t) \rangle dt > 0,$$

where as before $\varphi = \lambda(X)$. To show this let B be the compact d_c -ball of radius $\rho/2$ around p . Then there is $c \geq 1$ such that for all $u \in B$, all $W, \tilde{W} \in \mathbb{R}^m$, and $Z \in T_u M$

- (i) $\|d\theta_u(W, \tilde{W})\| \leq c\|W\|\|\tilde{W}\|$,
- (ii) $\|a_u(W, \tilde{W})\| \leq c\|W\|\|\tilde{W}\|$,
- (iii) $\|(Zd\theta)(W, \tilde{W})\| \leq c\|Z\|\|W\|\|\tilde{W}\|/2$.

Now if $Y \in HQ_\varphi$, then $Y(0) = 0$ and consequently $\theta Y(t) = \int_0^t \frac{d}{ds} \theta Y(s) ds$ for all $t \in I$. Since $\varphi(I) \subset B$ and $\frac{d}{ds} \theta Y(s) = (d\Theta_\varphi Y)(s) - 2d\theta(\varphi'(s), Y(s))$ it follows from (i) and $\|\theta\varphi'(s)\| = \rho/2 \leq 1$ for all $s \in I$ that

$$\int_0^t \left\| \frac{d}{ds} \theta Y(s) \right\| ds \leq \int_0^t \|(d\Theta_\varphi Y)(s)\| ds + c \int_0^t \|Y(s)\| ds.$$

For $s \leq t$ we have $\|Y(s)\| \leq \int_0^s \frac{d}{du} \theta Y(u) du$, hence

$$\int_0^t \left\| \frac{d}{ds} \theta Y(s) \right\| ds \leq \int_0^t \|(d\Theta_\varphi Y)(s)\| ds + ct \int_0^t \left\| \frac{d}{ds} \theta Y(s) \right\| ds.$$

Thus if $t \leq 1/c$, then

$$\begin{aligned} \|Y(t)\|^2 &\leq \left(\int_0^t \left\| \frac{d}{ds} \theta Y(s) \right\| ds \right)^2 \\ &\leq (1-ct)^{-2} \left(\int_0^t \|(d\Theta_\varphi Y)(s)\| ds \right)^2 \\ &\leq \frac{t}{(1-ct)^2} \int_0^t \|(d\Theta_\varphi Y)(s)\|^2 ds. \end{aligned}$$

Now (ii) and (2) for $\theta\tilde{\lambda}(X)$ show $\|\frac{d}{dt}\tilde{\lambda}(X)(t)\| \leq c\|\tilde{\lambda}(X)(t)\|$ and consequently $\|\tilde{\lambda}(X)(t)\| \leq e^{ct}\|X\|$. Thus for $\delta < 1/c$ we obtain

$$\begin{aligned} &\left| \int_0^\delta \langle (d\Theta_\varphi Y)(t), ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt \right| \\ &\leq \left(\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2 dt \right)^{1/2} \left(\int_0^\delta \|ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t))\|^2 dt \right)^{1/2} \\ &\leq ce^{c\delta}\|X\| \left(\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2 dt \right)^{1/2} \left(\int_0^\delta \|Y(t)\|^2 dt \right)^{1/2}. \end{aligned}$$

Inserting $\|Y(t)\|^2 \leq (\delta/(1-c\delta)^2) \int_0^\delta \|(d\Theta_\varphi Y)(s)\| ds$ ($t \leq \delta$), this yields

$$\begin{aligned} &\left| \int_0^\delta \langle (d\Theta_\varphi Y)(t), ad^*(\theta Y(t), \theta\tilde{\lambda}(X)(t)) \rangle dt \right| \\ &\leq (\delta/(1-c\delta)) ce^{c\delta}\|X\| \int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2 dt. \end{aligned}$$

On the other hand it follows from (iii) that

$$\begin{aligned} &\left| \int_0^\delta 2 \langle (Y(t) d\theta)(\varphi'(t), Y(t)), \theta\tilde{\lambda}(X)(t) \rangle dt \right| \\ &\leq c \left(\int_0^\delta \|Y(t)\|^2 dt \right) \left(\int_0^\delta \|\tilde{\lambda}(X)(t)\| dt \right) \\ &\leq (\delta^3/(1-c\delta)^2) ce^{c\delta}\|X\| \int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2 dt. \end{aligned}$$

Thus if we choose $\delta > 0$ sufficiently small that

$$\delta/(1-c\delta) < \min\{1, (2c\|X\|)^{-1}\},$$

then $I_\delta(Y, Y)$ is not smaller than a positive multiple of $\int_0^\delta \|(d\Theta_\varphi Y)(t)\|^2 dt$ which is positive for all $Y \in HQ_\varphi$ not vanishing identically on $[0, \delta]$. This is the claim.

Corollary 5.8. \exp_p^c is of maximal rank on an open and dense subset of $\{X \in T_p M \mid \|X\| < \kappa\}$.

Proof. We argue by contradiction and assume that there is an open subset U of $\{X \in T_p M \mid \|X\| < \kappa\}$ such that \exp_p^c is singular at every $X \in U$. By ev. diminishing the size of U we may assume that the rank of \exp_p^c is constant on U and that $\exp_p^c U$ is a smooth embedded submanifold N of M of dimension $n = \text{rank } \exp_p^c|_U < m$. For $X \in U$ the tangent space of N at $u = \exp_p^c X$ equals the vector space of all endpoints of Jacobi fields along $\lambda(X)$, which vanish at $p = \lambda(X)(0)$, and by 5.6, 5.7, and the choice of U this space is just $\{Y(1) \mid Y \in HQ_{\lambda(X)}\}$. This means in particular that Q_u is contained in $T_u N$ for all $u \in N$. Hence for all $u \in N$, $T_u N$ contains the span at u of the Lie algebra generated by Q_u . This span is $T_u M$ since Q is bracket generating which implies the contradiction $m > \dim N = \dim T_u N \geq \dim T_u M = m$.

Corollary 5.9. Every critical point of E is a geodesic with respect to d_c .

Proof. Let $X \in T_p M$ with $\|X\| = \kappa$, let $\varphi = \lambda(X)$, and assume $\text{rank } R_\varphi = n \leq m$. Then $W = \{d\Theta_\varphi Y(1)_{\varphi(1)} \mid Y \in J(\varphi), PY = 0\}$ is an $(m - n)$ -dimensional subspace of $T_{\varphi(1)} M$.

Choose a smooth $(m - n)$ -dimensional submanifold S of U containing $\varphi(1)$ with the property that W is the tangent space of S at $\varphi(1)$. Then $\Lambda = \{\psi \in H_1^p(I, U) \mid \psi(1) \in S\}$ is a smooth submanifold of $H_1^p(I, U)$. By Remark 2.7 for every $\psi \in \Lambda$ the g_ψ -orthogonal complement of the tangent space Λ_ψ of Λ at ψ consists of all $Y \in J(\psi)$ with $d\Theta_\psi Y(1)_{\psi(1)} \in (TS)^\perp$. By the definition of W this means that HQ meets Λ transversally at φ . Moreover the g_φ -orthogonal complement of the intersection $\Lambda_\varphi \cap HQ_\varphi$ in HQ_φ is just $\Omega(\varphi)^\perp$. Thus there is an open neighborhood A of φ in HQ such that $A \cap \Lambda$ is a smooth submanifold of HQ and φ is a critical point of the restriction of E to $A \cap \Lambda$.

By Lemma 5.7 the Hessian of $E|_{A \cap \Lambda}$ at φ is positive definite. Hence there is an open neighborhood B of φ in $A \cap \Lambda$ such that $E(\psi) > E(\varphi)$ for all $\psi \in B$, say B is the intersection of $A \cap \Lambda$ with the preimage under Θ of a 2ε -neighborhood of $\theta\varphi'$ in the Hilbert space $H_0(I, Q)$ for some $\varepsilon > 0$.

Assume that there is $\tau \in [0, 1 - \varepsilon]$ such that $d_c(\varphi(\tau), \varphi(\tau + \varepsilon)) < \|\tilde{P}X\|\varepsilon$. Let $\psi: [\tau, \tau + \varepsilon] \rightarrow U$ be a minimizing geodesic joining $\psi(\tau) = \varphi(\tau)$ to $\psi(\tau + \varepsilon) = \varphi(\tau + \varepsilon)$. Then the curve

$$t \rightarrow \begin{cases} \varphi(t) & \text{if } t \in [0, \tau] \cup [\tau + \varepsilon, 1], \\ \psi(t) & \text{if } t \in [\tau, \tau + \varepsilon] \end{cases}$$

is contained in B (recall $\|X\| = \kappa \leq 1$) and its energy is strictly smaller than $E(\varphi)$. This is a contradiction.

6. Isometries

In this section we investigate isometries of CC-metrics and show that they are necessarily smooth maps. Let $f: (M, d_c) \rightarrow (\tilde{M}, \tilde{d}_c)$ be an isometry. Then f is a homeomorphism which maps the space of H_1 -curves in M which are tangent almost everywhere to Q onto the space of H_1 -curves in \tilde{M} which are tangent almost everywhere to the distribution \tilde{Q} inducing \tilde{d}_c .

Let $p \in M$ and let U be an open neighborhood of p in M such that $TM|_U$ and $T\tilde{M}|_{f(U)}$ admit admissible trivializations X^1, \dots, X^m and $\tilde{X}^1, \dots, \tilde{X}^m$ as before. Then the assignment $\varphi \rightarrow f \circ \varphi$ is a bijection of $HQ = \{\varphi \in H_1^p(I, U) \mid \varphi'(t) \in Q \text{ for almost all } t \in I\}$ onto

$$H\tilde{Q} = \{\varphi \in H_1^{f(p)}(I, f(U)) \mid \varphi'(t) \in \tilde{Q} \text{ for almost all } t \in I\}.$$

Now if $\varphi: I \rightarrow U$ is an element of HQ , then φ is rectifiable with respect to d_c , and moreover $\|\varphi'(t)\|$ equals the dilation of φ at t for almost every $t \in I$, i.e.,

$$\|\varphi'(t)\| = \limsup_{\varepsilon \rightarrow 0} \frac{d_c(\varphi(t), \varphi(t + \varepsilon))}{\varepsilon}$$

(see [14, 17]). Since f is an isometry, this means $\|(f\varphi)'(t)\|_{\tilde{Q}} = \|\varphi'(t)\|$ for almost every $t \in I$, i.e., the map $\varphi \rightarrow f \circ \varphi$ commutes with the energy function.

The above trivialization of TM on U gives rise to an exponential map \exp_p^c at p as before, which is defined on an open star-shaped neighborhood W of 0 in $T_p M$. In the same way an exponential map $\exp_{f(p)}^c$ at $f(p)$ is defined on an open neighborhood \tilde{W} of 0 in $T_{f(p)} \tilde{M}$. For $X \in W$, define $\lambda(X) \in HQ$ by $\lambda(X)(t) = \exp_p^c tX$ ($t \in I$), i.e., we assume in the sequel always that $\lambda(X)$ is parametrized on I . It follows from Corollary 5.9 that for every $X \in W$ there is $\tilde{X} \in \tilde{W}$ such that $F\lambda(X) = f \circ \lambda(X) = \lambda(\tilde{X})$, i.e., the map $F: \lambda(X) \rightarrow f \circ \lambda(X)$ is a bijection of the space of geodesics in U , which emanate from p and are parametrized on I onto the space of geodesics in $f(U)$ which emanate from $f(p)$ and are parametrized on I . Notice that $\tilde{X} \in T_{f(p)} \tilde{M}$ with $\lambda(\tilde{X}) = F\lambda(X)$ is not necessarily unique.

Lemma 6.1. *Let $X \in T_p M$, $\tilde{X} \in T_{f(p)} \tilde{M}$, and $F\lambda(X) = \lambda(\tilde{X})$. If \exp_p^c is of maximal rank at X and $\exp_{f(p)}^c$ is of maximal rank at \tilde{X} , then:*

- (i) $f \exp_p^c tX = \exp_{f(p)}^c t\tilde{X}$ for all t for which $\exp_p^c tX$ is defined,
- (ii) *there is an open neighborhood Ω of X in $T_p M$, which is mapped by \exp_p^c diffeomorphically into U , and a diffeomorphism Ψ of $f(\exp_p^c \Omega)$ into $T_{f(p)} \tilde{M}$ such that $F\lambda(Y) = \lambda(\Psi \circ f \circ \exp_p^c Y)$ for all $Y \in \Omega$.*

Proof. (i) Assume that there is $\nu > 1$ such that $f \exp_p^c \nu X \neq \exp_{f(p)}^c \nu \tilde{X}$. Since $t \rightarrow f \exp_p^c tX$ ($t \in [0, \nu]$) is a geodesic in \tilde{M} there is $Y \in T_{f(p)} \tilde{M}$ such that $f \exp_p^c tX = \exp_{f(p)}^c tY$ for all $t \in [0, \nu]$. Now $\exp_{f(p)}^c \nu Y \neq \exp_{f(p)}^c \nu \tilde{X}$ shows $\tilde{X} \neq Y$. On the other hand $\lambda(Y) = F\lambda(X) = \lambda(\tilde{X})$ which means $\tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} Y = \tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} \tilde{X}$ (compare Lemma 2.6) or $\tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} (Y - \tilde{X}) = 0$ in contradiction to $\text{rank } R_{\lambda(\tilde{X})} = m$.

(ii) By the assumptions on X and \tilde{X} there are open neighborhoods B of X in $T_p M$ and \tilde{B} of \tilde{X} in $T_{f(p)} \tilde{M}$ with the following properties:

- (i) \exp_p^c maps B diffeomorphically onto an open neighborhood A of u in U .
- (ii) There is a diffeomorphism $\tilde{\Psi}$ of fA onto \tilde{B} such that $\exp_{f(p)}^c \circ \tilde{\Psi} = \text{Id}_{fA}$.

Assume that \tilde{B} contains the 2δ -neighborhood of \tilde{X} in $T_{f(p)} \tilde{M}$. If the lemma does not hold, then there is a sequence $\{X_k\} \subset B$ with $X_k \rightarrow X$ ($k \rightarrow \infty$) such that $F\lambda(X_k) = \lambda(Y_k)$ for some $Y_k \in T_{f(p)} \tilde{M}$ with $\|Y_k - \tilde{X}\| > \delta$ for all $k > 0$. Since $d_\infty(\lambda(\tilde{X}), \lambda(Y_k)) \rightarrow 0$ ($k \rightarrow \infty$), Lemma 4.2 shows that the sequence $\{Y_k\} \subset T_{f(p)} \tilde{M}$ is bounded; hence passing to a subsequence we may assume that $\{Y_k\}_k$ converges to some $Y \in T_{f(p)} \tilde{M}$. Clearly $\|Y - \tilde{X}\| \geq \delta$. Since $d_\infty(\lambda(Y_k), \lambda(Y)) \rightarrow 0$ ($k \rightarrow \infty$), it follows that $\lambda(Y) = \lambda(\tilde{X})$. But this means $\tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} Y = \tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} \tilde{X}$ or $\tilde{P}(\text{Id} - T_{\lambda(\tilde{X})})^{-1} (Y - \tilde{X}) = 0$ in contradiction to $\text{rank } R_{\lambda(\tilde{X})} = m$. This shows the claim. q.e.d.

Now we are ready to show

Theorem 6.2. *An isometry $(M, d_c) \rightarrow (\tilde{M}, \tilde{d}_c)$ is smooth.*

Proof. We show first that an isometry $f: (M, d_c) \rightarrow (\tilde{M}, \tilde{d}_c)$ is smooth on an open dense subset of M . This follows from a successive application of the following.

Sublemma. *Let $N \subset M$ be a smooth embedded submanifold of dimension $n < m$. Assume that the restriction of f to N is smooth. Then for*

every point p of an open dense subset of N there is an open neighborhood U of p in N and a smooth $(n + 1)$ -dimensional embedded submanifold \bar{N} of M containing U such that the restriction of f to \bar{N} is smooth.

To show the sublemma observe that since Q is bracket generating, N contains an open dense subset with the property that for every p of this set the tangent space $T_p N$ of N at p does not contain Q_p .

Since, as an isometry of (M, d_c) onto (\tilde{M}, \tilde{d}_c) , f is absolutely continuous with respect to Lebesgue measure (compare [12]), there is $Y \in T_p M$ such that $\tilde{P}Y$ is transversal to N at p , $u = \lambda(Y)(1)$ is a regular value for \exp_p^c , and $f(u)$ is a regular value for $\exp_{f(p)}^c$. Let $X \in T_u M$ be such that $\lambda(X)(t) = \lambda(Y)(1 - t)$. By Remark 5.6 p is a regular value for \exp_u^c , and $f(p)$ is a regular value for $\exp_{f(u)}^c$. Choose $\Omega \subset T_u M$ and $\Psi: f(\exp_u^c \Omega) \rightarrow T_{f(u)} \tilde{M}$ as in Lemma 6.1. Then $\Omega \cap (\exp_u^c)^{-1}(N) = W$ is a smooth submanifold of Ω . Since the restriction of f to N is smooth, the same is true for the restriction of $Df = \Psi \circ f \circ \exp_u^c$ to W . Since $\lambda(X)$ meets N transversally at $\lambda(X)(1)$, there is an open neighborhood B of X in W and a number $\varepsilon > 0$ such that $\bar{N} = \{\exp_u^c tY \mid Y \in B, t \in (1 - \varepsilon, 1 + \varepsilon)\}$ is a smooth embedded submanifold of M . But $f \exp_u^c tY = \exp_{f(u)}^c t(DfY)$ for all $Y \in B$ and $t \in (1 - \varepsilon, 1 + \varepsilon)$ shows that the restriction of f to \bar{N} is smooth. This finishes the proof of the sublemma.

To finish the proof of the theorem let $p \in M$ be arbitrary. Then there is a regular value $w \in M$ of \exp_p^c such that $f(w)$ is a regular value of $\exp_{f(p)}^c$ and f is smooth near w . Let $\lambda(Y)$ ($Y \in T_p M$) be a minimizing geodesic joining p to w , and let Ω and Ψ be as in Lemma 6.1. If we choose Ω sufficiently small, then $Df = \Psi \circ f \circ \exp_p^c$ is a diffeomorphism of Ω into $T_{f(p)} \tilde{M}$ such that $F\lambda(Z) = \lambda(DfZ)$ for all $Z \in \Omega$. Lemma 6.1 shows $f \exp_p^c tZ = \exp_{f(p)}^c t(DfZ)$ for all $Z \in \Omega$ and all t for which both sides are defined.

For $Z \in \Omega$, define $\alpha(Z) = -\tilde{\lambda}(Z)(1)$ and $\beta(Z) = -\tilde{\lambda}(DfZ)(1)$, where $\tilde{\lambda}(Z)$ is as before. Then α and β are smooth maps of Ω into TM and $T\tilde{M}$, resp., and $\exp_{\lambda(Z)(1)}^c$ is of maximal rank at $\alpha(Z)$ and $\exp_{f\lambda(Z)(1)}^c$ is of maximal rank at $\beta(Z)$. Hence there is a compact neighborhood K of Y in Ω and $\varepsilon > 0$ such that, for every $Z \in K$, $\exp_{\lambda(Z)(1)}^c$ and $\exp_{f\lambda(Z)(1)}^c$ are of maximal rank at $(1 + \varepsilon)\alpha(Z)$ and $(1 + \varepsilon)\beta(Z)$ respectively.

Since $\{\varepsilon Z \mid -Z \in K\}$ has nonempty interior, Corollary 5.8 shows that there is $W \in K$ such that $\lambda(-\varepsilon W)(1) = u$ is a regular value for \exp_p^c , and $f(u)$ is a regular value for $\exp_{f(p)}^c$.

Let $X = -(1 + \varepsilon)\tilde{\lambda}(W)(-\varepsilon)$ and $\tilde{X} = -(1 + \varepsilon)\tilde{\lambda}(DfW)(-\varepsilon)$. We have $\lambda(X)(t) = \exp_p^c((1 + \varepsilon)t - \varepsilon)W$ and $\lambda(\tilde{X})(t) = \exp_{f(p)}^c((1 + \varepsilon)t - \varepsilon)(DfW)$, and consequently $F\lambda(X) = \lambda(\tilde{X})$.

By Lemma 6.1 there is an open neighborhood B of X in T_uM and a diffeomorphism Φ of B into $T_{f(u)}\tilde{M}$ such that $F\lambda(Z) = \lambda(\Phi Z)$ for all $Z \in B$. Since \exp_u^c is of maximal rank at $\varepsilon X/(1 + \varepsilon)$, for sufficiently small B the map $Z \rightarrow \lambda(Z)(\varepsilon/(1 + \varepsilon))$ is a diffeomorphism of B onto an open neighborhood of p . But $f\lambda(Z)(t) = \lambda(\Phi Z)(t)$ for all $t \in I$ and $Z \in B$ then implies that f is smooth near p . Hence the proof of the theorem is finished.

7. The strong bracket generating case

In this section we investigate the group of isometries of a CC-metric d_c which is induced by a distribution Q satisfying the strong bracket generating hypothesis (see [17]), i.e., for every nonzero section X of Q , TM is generated by Q and $[X, Q]$.

Let N be the annihilator of Q in the cotangent bundle T^*M of M . N is a smooth $k = (m - q)$ -dimensional subbundle of T^*M .

Lemma 7.1. *Every Riemannian metric on Q gives rise to a unique Riemannian metric on N .*

Proof. Let $p \in M$, $0 \neq \omega_p \in N_p$, and let ω be a local section of N through ω_p . If X, Y are local sections of Q near p , then

$$d\omega(X, Y) = \frac{1}{2}\{X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])\} = -\frac{1}{2}\omega([X, Y]),$$

in particular $d\omega(X_p, Y_p)$ only depends on ω_p , not on the choice of the extension ω . Since the commutators of sections of Q span TM , the restriction of $d\omega$ to Q_p does not vanish. This means that there is a natural injective bundle map J of N into the exterior product $Q^* \wedge Q^*$. Since a Riemannian metric on Q induces a Riemannian metric on $Q^* \wedge Q^*$, this metric can be pulled back via J to a metric on N . q.e.d.

For a section ω of N , define a map $J\omega: Q \rightarrow T^*M$ by $(J\omega(X))(Y) = d\omega(X, Y)$. Since Q satisfies the strong bracket generating hypothesis, $J\omega$ is an injective bundle map and $J\omega(Q)$ is complementary to N .

Lemma 7.2. *Let $\omega^1, \dots, \omega^k$ be a local orthonormal basis of N with respect to the metric of 7.1. Then for every $i \in \{1, \dots, k\}$ the $(m - 1)$ -dimensional subspace of T_p^*M , which is spanned by $J\omega^i(Q_p) \cup \{\omega_p^j | i \neq j\}$, only depends on $\omega_p^1, \dots, \omega_p^k$ and is transversal to $\{\lambda\omega_p^i | \lambda \in \mathbb{R}\}$.*

Proof. Clearly $A_p^i = \text{span}(J\omega^i(Q_p) \cup \{\omega_p^j | i \neq j\})$ is transversal to ω_p^i . To show that A_p^i only depends on $\omega_p^1, \dots, \omega_p^k$ let $\bar{\omega}^1, \dots, \bar{\omega}^k$ be another local orthonormal basis of N near p with $\bar{\omega}_p^i = \omega_p^i$. Then there is a smooth function (g_{ij}) of a neighborhood of p in M into the special orthonormal group $\text{SO}(k)$ such that $(g_{ij})(p) = \text{Id}$ and $\bar{\omega}^i = \sum_j g_{ij} \omega^j$. We have $(dg_{jj})_p = 0$ for $j = 1, \dots, n$. Let X^1, \dots, X^q be a local orthonormal basis of Q near p and let $\sigma^j = J\omega^i(X^j)$. Then $dg_{ij} = \sum_\alpha a_\alpha^{ij} \sigma^\alpha + \sum_\beta b_\beta^{ij} \omega^\beta$ with smooth functions a_α^{ij} and b_β^{ij} , and

$$\begin{aligned} d\bar{\omega}^i &= \sum_j dg_{ij} \wedge \omega^j + \sum_j g_{ij} d\omega^j \\ &= \sum_{\alpha, j} a_\alpha^{ij} \sigma^\alpha \wedge \omega^j + \sum_{\beta, j} b_\beta^{ij} \omega^\beta \wedge \omega^j + \sum_j g_{ij} d\omega^j. \end{aligned}$$

Since $a_\alpha^{jj} = 0$ for $j = 1, \dots, n$, this implies $(J\bar{\omega}^i)(X) = (J\omega^i)(X) + \sum_{i \neq j} \sum_\alpha a_\alpha^{ij}(X) \omega^j$ for every $X \in Q_p$, i.e., $(J\bar{\omega}^i)(X) - (J\omega^i)(X) \in \text{span}\{\omega^j | i \neq j\}$ as claimed.

Corollary 7.3. *If Q satisfies the strong bracket generating hypothesis, then every Riemannian metric $\langle \cdot, \cdot \rangle_Q$ on Q can intrinsically be extended to a Riemannian metric on M .*

Proof. Let $p \in M$. By 7.2 the choice of an orthonormal basis $\omega_p^1, \dots, \omega_p^k$ of N_p determines for every $i \in \{1, \dots, k\}$ an $(m-1)$ -dimensional subspace of T_p^*M , which annihilates a 1-dimensional subspace A^i of T_pM transversal to the kernel of ω_p^i . Let $Z^i \in A^i$ be such that $\omega_p^i(Z^i) = 1$.

The vectors Z^1, \dots, Z^k span a k -dimensional subspace of T_pM which is complementary to Q_p ; we thus can define an extension $g(\omega_p^1, \dots, \omega_p^k)$ of $(\langle \cdot, \cdot \rangle_Q)_p$ by choosing the vectors Z^1, \dots, Z^k orthonormal and perpendicular to Q_p .

Now the space of orthonormal bases of N_p can be identified with the orthogonal group $\text{O}(k)$. Let μ be the normalized Haar measure on $\text{O}(k)$ (which satisfies $\mu(\text{O}(k)) = 1$), and define $\langle X, Y \rangle_p = \int_{\text{O}(k)} g(\xi)(X, Y) d\mu(\xi)$ for $X, Y \in T_pM$. Then $\langle \cdot, \cdot \rangle_p$ is a scalar product on T_pM extending the product on Q_p and moreover is defined intrinsically by $(Q, \langle \cdot, \cdot \rangle_Q)$. q.e.d.

The Riemannian metric $\langle \cdot, \cdot \rangle$ on M defined in 7.3 will be called the *canonical extension* of $\langle \cdot, \cdot \rangle_Q$.

Since every isometry of d_c is smooth, 7.3 yields

Corollary 7.4. *The group of isometries of d_c is a closed subgroup of the Lie group of isometries of the canonical extension of $\langle \cdot, \cdot \rangle_Q$.*

Lemma 7.5.

$$\begin{aligned} \theta \frac{D}{dt} \lambda(X)'(t)|_{t=0} &= \tilde{P} ad^*(\tilde{P}\theta X, (1 - \tilde{P})\theta X) \\ &\quad - (1 - \tilde{P}) ad^*(\tilde{P}\theta X, \tilde{P}\theta X). \end{aligned}$$

Proof. Let $\varphi = \lambda(X)$. Then $\theta\varphi'(0) = \tilde{P}\theta X$, $\frac{d}{dt}\theta\varphi'(t)|_{t=0} = \tilde{P} ad^*(\tilde{P}\theta X, \theta X)$, and, moreover by Lemma 2.3, $\theta \frac{D}{dt} \varphi'(t) = \frac{d}{dt} \theta \varphi'(t) - ad^*(\theta\varphi'(t), \theta\varphi'(t))$. Together this yields the claim.

Corollary 7.6. *If Q satisfies the strong bracket generating hypothesis, then every d_c -geodesic through p is uniquely determined by its tangent and its covariant derivative at p . In particular \exp_p^c is intrinsically defined.*

Proof. For $Y \in Q_p$ the map $\alpha_Y: Q_p^\perp \rightarrow Q_p, Z \rightarrow \tilde{P}a_p(\theta Y, \theta Z)$ does not depend on the choice of the local trivialization of TM near p , and is injective. The corollary thus follows from 7.5 and the fact that every Riemannian metric on a strong bracket generating distribution can intrinsically be extended to a Riemannian metric on M .

8. Nilpotent homogeneous Lie groups

In this section we investigate the group of isometries of a left-invariant CC-metric d_c on a nilpotent homogeneous Lie group. Thus let N be a nilpotent homogeneous simply connected Lie group whose Lie algebra \mathfrak{M} is generated by a complement Q of its derived algebra $[\mathfrak{M}, \mathfrak{M}]$, i.e., if $Q^1 = Q$ and $Q^{i+1} = [Q, Q^i]$, then there is $k \geq 1$ such that $\mathfrak{M} = \bigoplus_{i=1}^k Q^i$ (direct sum). For every $r > 0$ the assignment $\delta_r: \sum_{i=1}^k X^i \rightarrow \sum_{i=1}^k r^i X^i$ ($X^i \in Q^i$) is a Lie algebra automorphism of \mathfrak{M} which integrates to an automorphism Δ_r of N . Let d_c be a left-invariant CC-metric on N induced by a scalar product $\langle \cdot, \cdot \rangle_Q$ on Q . Then $\{\Delta_t | t > 0\}$ is a 1-parameter group of homotheties with respect to d_c , i.e., $d_c(\Delta_r p, \Delta_r u) = r d_c(p, u)$ for all $p, u \in N, r > 0$ (compare [13]).

Choose an extension of $\langle \cdot, \cdot \rangle_Q$ to a scalar product $\langle \cdot, \cdot \rangle$ on \mathfrak{M} such that the decomposition $\mathfrak{M} = \bigoplus_{i=1}^k Q^i$ is $\langle \cdot, \cdot \rangle$ -orthogonal and a left-invariant $\langle \cdot, \cdot \rangle$ -orthonormal trivialization of TN . Denote as before by $\lambda(X)$ ($X \in \mathfrak{M}$) the geodesic through the identity $\lambda(X)(0) = e$ with respect to these data. Since $\theta \frac{d}{ds} (\Delta_t \lambda(X)(s)) = t \theta (\frac{d}{ds} \lambda(X)(s))$ for all $t > 0$ and $s \in \mathbb{R}$, we have $\Delta_t \lambda(X) = \lambda(t^2 \delta_{1/t} X)$.

Lemma 8.1. *Every isometry of (N, d_c) fixing the identity e permutes the 1-parameter subgroups of N which are tangent to Q .*

Proof. For every $X \in Q$ the geodesic $\lambda(X)$ is the 1-parameter subgroup in N defined by X and is globally minimizing since the 1-parameter subgroups tangent to Q are globally minimizing geodesics with respect to the Riemannian metric $\langle \cdot, \cdot \rangle$ on N . Since f is smooth, $f\lambda(X)$ is a globally minimizing geodesic in (N, d_c) with $\text{rank } R_{f\lambda(X)} = \text{rank } R_{\lambda(X)} = k$ for some $k \leq m$.

For $Y \in Q$, define $\text{rank}(Y)$ to be the dimension of the smallest ad Y -invariant subspace of \mathfrak{M} containing Q . Remark 2.8 shows $\text{rank } R_\varphi \geq \text{rank}(\varphi'(0))$ for every smooth curve $\varphi: I \rightarrow N$ through $\varphi(0) = e$ which is tangent to Q , and moreover the rank is preserved by every diffeomorphism of N which leaves Q invariant and fixes e . If φ is a 1-parameter subgroup of N tangent to Q then $\text{rank } R_\varphi = \text{rank}(\varphi'(0))$.

For every $r > 0$, $f(r) = \Delta_r f \Delta_{1/r}$ is an isometry of (N, d_c) fixing e . Let B be the compact d_c -ball of radius $2\|X\|$ around e . By Ascoli's theorem there is a sequence r_i ($i > 0$) such that $r_i \rightarrow 0$ ($i \rightarrow \infty$) and that the sequence of maps $f(r_i)$ converges uniformly on B to a map \bar{f} . Since \bar{f} is an isometry of (B, d_c) fixing e , by 7.2 a diffeomorphism of B , and $\bar{f}\lambda(X)$ is a minimizing geodesic in (N, d_c) with $\text{rank } R_{\bar{f}\lambda(X)} = \text{rank}(\bar{f}\lambda(X)'(0)) = k$.

Let $A \subset \mathfrak{M}$ be the $\langle \cdot, \cdot \rangle$ -orthogonal complement in \mathfrak{M} of $\mathfrak{M} \cap (\text{Id} - T_{\bar{f}\lambda(X)})L^\infty(I, Q^\perp)$, and let $\bar{P}: \mathfrak{M} \rightarrow A$ be the $\langle \cdot, \cdot \rangle$ -orthogonal projection. Since $\text{rank } R_{\bar{f}\lambda(X)} = \text{rank}(\bar{f}\lambda(X)'(0))$, Remark 2.8 shows that A equals the smallest ad $(\bar{f}\lambda(X)'(0))$ -invariant subspace of \mathfrak{M} containing Q and hence is invariant under the automorphisms δ_r ($r > 0$). By 3.2, A is transversal to $\mathfrak{M} \cap (\text{Id} - T_{f(r_i)\lambda(X)})L^\infty(I, Q^\perp)$ for all sufficiently large $i > 0$; we may assume that this is true for all i . This means that there is a unique $Y \in A$ such that $f\lambda(X) = \lambda(Y)$. Now $f(r)\lambda(X) = \lambda(r\delta_{1/r}Y)$ for all $r > 0$, and since A is invariant under the automorphisms δ_r ($r > 0$), we have $\|\bar{P}(r\delta_{1/r}Y)\| = \|r\delta_{1/r}Y\|$, and hence by Remark 4.3 the sequence $\|r_i\delta_{1/r_i}Y\|$ is uniformly bounded. But if $Y = Y_1 + Y_2$ with $Y_1 \in Q$ and $Y_2 \in Q^\perp$, we have $\|r\delta_{1/r}Y\| \geq \|Y_2\|/r$ for all $r > 0$ and consequently since $r_i \rightarrow 0$ necessarily $Y_2 = 0$, i.e., $\lambda(Y)$ is a 1-parameter subgroup in N as claimed. q.e.d.

For $i \geq 1$, denote by H_i the Lie subgroup of N whose Lie algebra is the ideal $\mathfrak{h}_i = \bigoplus_{j=i}^k Q^j$. $\langle \cdot, \cdot \rangle_Q$ induces a left-invariant CC-metric d_i

on the factor group N/H_i in such a way that the canonical projection $\pi_i: (N, d_c) \rightarrow (N/H_i, d_i)$ is distance decreasing.

Let \exp be the exponential map of N and call two pairs $(u, X), (v, Y) \in N \times Q$ parallel if the function $t \rightarrow d_c(u \exp tX, v \exp tY)$ is bounded on \mathbb{R} .

Lemma 8.2. *If (u, X) and (v, Y) are parallel, then $X = Y$.*

Proof. Let Ψ be the restriction to Q of the map $\pi_2 \circ \exp$. Ψ is an isometry of $(Q, \langle \cdot, \cdot \rangle_Q)$ onto $(N/H_2, d_2)$. The Campbell-Hausdorff formula [5] shows $\pi_2(u \exp tX) = \Psi((\Psi^{-1}\pi_2u) + tX)$ for all $u \in N$ and $X \in Q$, and hence

$$d_2(\pi_2(u \exp tX), \pi_2(v \exp tY)) \\ = \|(\Psi^{-1}\pi_2u - \Psi^{-1}\pi_2v) + t(X - Y)\| \geq |t| \|X - Y\| - \text{const.}$$

Since the latter expression is uniformly bounded for all $t \in \mathbb{R}$ whenever (u, X) and (v, Y) are parallel, the lemma follows. q.e.d.

Let Z be the center of N . Then we have

Lemma 8.3. *$p \in Z$ if and only if (p, X) is parallel to (e, X) for all $X \in Q$.*

Proof. If $p \in Z$ then $d_c(\exp tX, p \exp tX) = d_c(e, p)$ for all $t \in \mathbb{R}$, i.e., (p, X) is parallel to (e, X) for all $X \in Q$.

On the other hand, if $u = \exp(\sum_{i=1}^k X^i) \notin Z (X^i \in Q^i)$, then there is $X \in Q$ such that $[X, \sum_{i=1}^k X^i] \neq 0$. Let $j - 2 = \min\{i \geq 1 \mid [X, X^i] \neq 0\}$ and let Ψ be the restriction to $S = \bigoplus_{i=1}^{j-1} Q^i$ of the map $\pi_j \circ \exp$. Under the identification of S with $\mathfrak{M}/\mathfrak{h}_j$, Ψ can be viewed as the exponential map of N/H_j , i.e., group multiplication in N/H_j can be computed via the Campbell-Hausdorff formula in S . This means

$$d_j(\pi_j \exp tX, \pi_j(u \exp tX)) = d_j(\pi_j e, \Psi((\Psi^{-1}\pi_j u) + t[X^j, X])),$$

which is unbounded in $t \in \mathbb{R}$. Since π_j is distance-decreasing, (u, X) is not parallel to (e, X) . q.e.d.

A special case of the following corollary is due to Pansu [13, Proposition 18.5]:

Corollary 8.4. *Every isometry of (N, d_c) fixing the identity e is a Lie group automorphism of N .*

Proof. By a theorem of Pansu [13] there is an automorphism Ψ of N such that $d_e(\Psi \circ f)|_Q = \text{Id}|_Q$. Ψ is necessarily an isometry with respect to d_c ; hence we only have to show that there is no nontrivial isometry f of (N, d_c) with $f(e) = e$ and $d_e f|_Q = \text{Id}|_Q$.

We proceed by induction on the degree of nilpotency of N . If this degree equals 1, then (N, d_c) is Euclidian and hence there is nothing to show. Thus let $k \geq 2$ and assume the claim is known for all groups of degree $\leq k - 1$. Let N be a group of degree k and let f be as above. By 8.1, f permutes the integral curves of left-invariant vector fields and hence maps parallel elements (u, X) and (v, X) of $N \times Q$ onto parallel elements. Since $f(e) = e$ and $f \exp tX = \exp tX$ for all $X \in Q$, $t \in \mathbb{R}$, this implies by 8.2 and 8.3 the following:

- (i) f preserves the center Z of N ,
- (ii) $f(p \exp tX) = f(p) \exp tX$ for all $p \in Z$, $X \in Q$, and $t \in \mathbb{R}$.

For every $p \in Z$ the map $f_p: u \rightarrow f(p)^{-1}f(up)$ is an isometry of (N, d_c) fixing e which by (ii) satisfies $d_e f_p|_Q = \text{Id}|_Q$. By (i) f_p preserves the center Z of N and hence induces a transformation \bar{f}_p of the factor group N/Z which is an isometry with respect to the induced CC-metric. Since the degree of nilpotency of N/Z equals $k - 1$, by the induction hypothesis \bar{f}_p equals the identity of N/Z . This is true for every $p \in Z$. Thus the differential of f preserves the left-invariant vector fields tangent to Q . Since Q generates \mathfrak{M} , f is the identity. Hence the proof is finished. q.e.d.

We conclude this work with the following example.

Example 8.5. (a) The Lie algebra \mathfrak{h} of the 3-dimensional Heisenberg group H^3 is spanned by vectors X, Y , and Z which satisfy the relations $[X, Y] = Z$ and $[X, Z] = [Y, Z] = 0$. Let $\langle \cdot, \cdot \rangle$ be the scalar product on \mathfrak{h} for which this basis is orthonormal, and let d_c be the left-invariant Carnot-Carathéodory metric on H^3 induced by $Q = \text{span}\{X, Y\}$ and $\langle \cdot, \cdot \rangle_Q = \langle \cdot, \cdot \rangle_Q$. We want to compute the geodesics $\lambda(W): t \rightarrow \exp_e^c tW$ of d_c through the identity e (compare [10]). For $x_0, y_0, z_0 \in \mathbb{R}$ write $\theta \tilde{\lambda}(x_0 X + y_0 Y + z_0 Z)(t) = x(t)X + y(t)Y + z(t)Z$. By using the relations $(\text{ad } X)^* Z = Y$ and $(\text{ad } Y)^* Z = -X$ equation (2) of Lemma 4.1 transforms into the following system of differential equations for the coordinate functions x, y, z :

$$\begin{aligned} x'(t) &= -y(t)z(t), & x(0) &= x_0, \\ y'(t) &= x(t)z(t), & y(0) &= y_0, \\ z'(t) &= 0, & z(0) &= z_0. \end{aligned}$$

Hence

$$\begin{aligned} \theta \lambda(x_0 X + y_0 Y + z_0 Z)'(t) \\ = (x_0 \cos z_0 t - y_0 \sin z_0 t)X + (x_0 \sin z_0 t + y_0 \cos z_0 t)Y. \end{aligned}$$

The Lie group exponential map \exp of H^3 induces global coordinates on H^3 via the identification of $\exp(x_1X+x_2Y+x_3Z)$ with $(x_1, x_2, x_3) \in \mathbb{R}^3$. In these coordinates the vector fields X , Y , and Z are given by [14]

$$X = \frac{\partial}{\partial x_1} - \frac{1}{2}x_2 \frac{\partial}{\partial x_3}, \quad Y = \frac{\partial}{\partial x_2} + \frac{1}{2}x_1 \frac{\partial}{\partial x_3}, \quad Z = \frac{\partial}{\partial x_3}.$$

Thus for $z_0 \neq 0$ the geodesic $\lambda(X+z_0Z)$ has the coordinate representation

$$\lambda(X+z_0Z)(t) = z_0^{-1}(\sin z_0t, 1 - \cos z_0t, t/2 - \sin z_0t/2z_0),$$

in particular $\lambda(X+z_0Z)(2\pi/z_0) = (0, 0, \pi/z_0^2)$. For every $\alpha \in S^1 \sim [0, 2\pi]$ the isometry of $(\mathfrak{h}, \langle \cdot, \cdot \rangle)$ which fixes the center of \mathfrak{h} and acts as a rotation of angle α in the plane Q is an automorphism of \mathfrak{h} which integrates to an automorphism Ψ_α of H^3 . Ψ_α is an isometry with respect to d_c which maps for every $W \in \mathfrak{h}$ the geodesic $\lambda(W)$ onto $\lambda(d\Psi_\alpha W)$. Thus for every $s > 0$ there is a S^1 -family $\{\lambda(d\Psi_\alpha(X + \sqrt{\pi/s}Z)) \mid \alpha \in S^1\}$ of minimizing d_c -geodesics joining e to $\lambda(d\Psi_\alpha(X + \sqrt{\pi/s}Z))(2\sqrt{\pi s}) = (0, 0, s)$. In particular each of the d_c -geodesics $\{\lambda(W) \mid \|\dot{P}W\| = 1\}$ minimizes exactly on the interval $[0, 2\pi(\|W\|^2 - 1)^{-1/2}]$.

(b) Let $\bar{H} = H^3 \times H^3$ be the direct product of two copies of H^3 with Lie algebra $\bar{\mathfrak{h}} = \mathfrak{h} \times \mathfrak{h}$, equipped with the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle^0$ which is the product of the metrics $\langle \cdot, \cdot \rangle$ on \mathfrak{h} above. Let \bar{d}_c be induced by $(Q \times Q, \langle \cdot, \cdot \rangle^-|_{Q \times Q})$. Then the map $(H^3, d_c) \rightarrow (\bar{H}, \bar{d}_c)$, $u \rightarrow (u, e)$ is an isometric embedding. In particular for every $X \in Q$ the 1-parameter subgroup in \bar{H} , which is tangent to $(X, 0)$ at e , is a minimizing \bar{d}_c -geodesic in \bar{H} . However the rank of R along this geodesic equals 5, i.e., the exponential map \exp_c^c of (\bar{H}, \bar{d}_c) at e is singular along $\{(tX, 0) \in \bar{\mathfrak{h}} \mid X \in Q, t \in \mathbb{R}\}$.

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